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ON DIMENSION OF CONFIGURATIONS

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Abstract
The combinatorial (or abstract) configuration is an incidence structure, which can often be represented in Euclidean or real projective plane by a system of points and lines. B. Grünbaum introduced the concept of the dimension of a configuration as the largest dimensional space into which the configuration can be embedded to preserve the incidence. In this paper we examine the dimensions of some configurations and introduce the concept of the natural d-configuration.

1 Introduction

Definition 1. *The combinatorial (or abstract) configuration* $\mathcal{C}^c$ *of the type* (p_q, n_k) *is an incidence structure with sets* $\mathcal{P}$ *and* $\mathcal{B}$ *of objects, called marks and blocks, such that the following conditions hold:*

(1) $|\mathcal{P}| = p$*;*

(2) $|\mathcal{B}| = n$*;*

(3) each mark is incident with q *blocks;*

(4) each block is incident with k *marks;*

(5) two distinct marks are incident with at most one block.

Configuration tables are a common way of presenting abstract configurations. In such a table the marks of each block are listed in a column that represents the block.

Definition 2. *The geometric (or point-line) configuration* $\mathcal{C}^g$ *is a geometric incidence structure consisting of points and (straight) lines, in the simplest case in Euclidean or real projective plane, such that the Conditions (1)-(4) of the definition 1 hold. Note that in this case (5) is fulfilled automatically. The incidence between points and lines means that the point lies on the line or, equivalently, that the line passes through the point.*

In other words, a geometric configuration $\mathcal{C}^g$ is a family of points and lines such that, for positive integers p, q, n, k each of the p points is incident with precisely q of the lines, while each of these n lines is incident with precisely k of the points.

The geometric configurations are usually represented by diagrams where points are illustrated by dots and lines by straight lines.

Two configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same incidences provided their points and lines can be given such labels that a point and a line are incident in $\mathcal{C}_1$ if and only if they are incident in $\mathcal{C}_2$.

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If in a configuration(combinatorial or geometric) of type (p_q, n_k) we have $p = n$, then the equality $q = k$ also holds; then the notation (n_k) is used. We use the term balanced configuration by [6]. Given two configurations, $\mathcal{C}_1$ and $\mathcal{C}_2$ with pairs $(\mathcal{P}_1, \mathcal{B}_1)$ and $(\mathcal{P}_2, \mathcal{B}_2)$, respectively, we say that they are isomorphic if there is a bijection which sends $\mathcal{P}_1$ to $\mathcal{P}_2$ and $\mathcal{B}_1$ to $\mathcal{B}_2$ such that incidences are preserved.

It is easy to see that the two abstract configurations given by the configuration tables 1 and 2 are isomorphic to each other.

Table 1. Configuration table of abstract configuration $\mathcal{C}_1^c(7_3)$

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Table 2. Configuration table of abstract configuration $\mathcal{C}_2^c(7_3)$

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

This configuration is known as the Fano configuration; which was described by Gino Fano in 1891. The Fano configuration has no geometric realization, i.e. no geometric configuration is isomorphic to the abstract Fano configuration, but it has topological realization in figure 1.

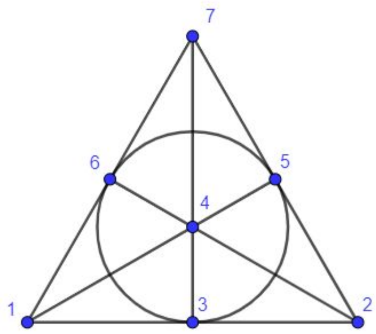
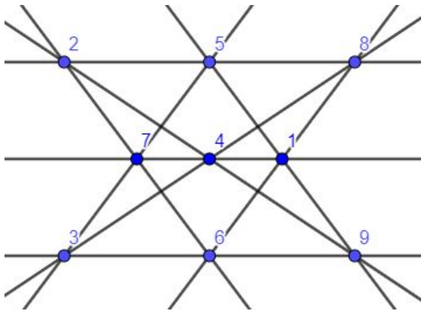


Figure 1. Topological configuration $\mathcal{C}(7_3)$

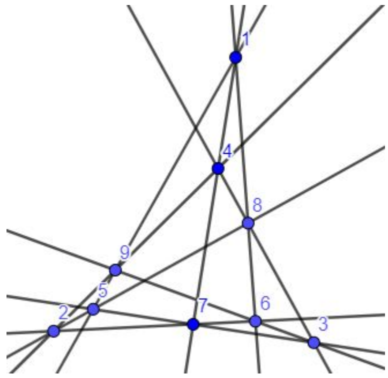
Configurations $\mathcal{C}^c(9_3)_1$, $\mathcal{C}_1^g(9_3)_1$ and $\mathcal{C}_2^g(9_3)_1$ are isomorphic (table 3, figure 2). In this case we say that $\mathcal{C}_1^g(9_3)_1$ and $\mathcal{C}_2^g(9_3)_1$ are different geometric realizations of the same abstract configuration $\mathcal{C}^c(9_3)_1$. Namely these two geometric configurations are essentially the same [6, 7].

Table 3. Configuration table of abstract configuration $\mathcal{C}^c(9_3)_1$

1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6
7	9	8	9	8	7	8	7	9



Geometric configuration $\mathcal{C}_1^g(9_3)_1$



Geometric configuration $\mathcal{C}_2^g(9_3)_1$

Figure 2. Isomorphic geometric configurations

Configurations $\mathcal{C}^c(9_3)_2$ and $\mathcal{C}^g(9_3)_2$ are isomorphic with each other (table 4, figure 3), but they are not isomorphic with configurations $\mathcal{C}^c(9_3)_1$, $\mathcal{C}_1^g(9_3)_1$ and $\mathcal{C}_2^g(9_3)_1$.

Table 4. Configuration table of abstract configuration $\mathcal{C}^c(9_3)_2$

1	1	1	2	2	2	3	3	4
3	4	5	4	5	6	5	6	7
7	6	8	8	7	9	9	8	9

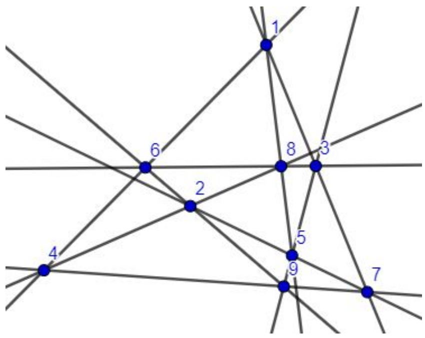


Figure 3. Geometric configuration $\mathcal{C}^g(9_3)_2$

2 The d-configurations

Instead of a plane, we examine the geometric realizations of the configurations in the d-dimensional (Euclidean, or projective) space [2, 3, 5], similar to graphs [4].

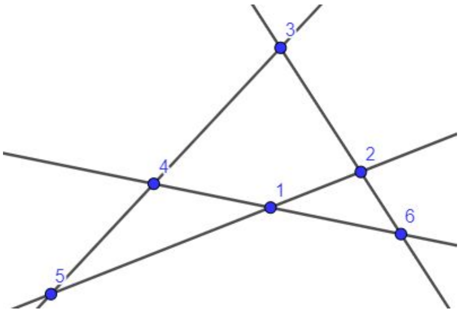
Definition 3. *The dimension of the configuration* $\mathcal{C}$ *is* d *if this is the largest integer for which configuration* $\mathcal{C}$ *has a geometric representation (by points and straight lines) in some Euclidean space, such that the affine hull of the imbedding has dimension d. The dimension of the configuration* $\mathcal{C}$ *is denoted by* $\dim(\mathcal{C})$.

We note that by a representation we mean a family of points and lines such that all the combinatorial incidences are satisfied but some points may be on lines with which they are not incident. In view of the above concepts, we introduce the notion of the d-configuration and the natural d-configuration:

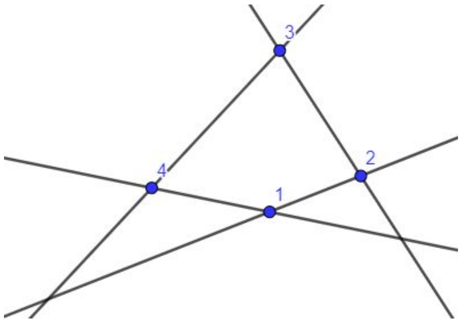
Definition 4. *A geometric realization of the configuration* $\mathcal{C}$ *(by points and straight lines) in d-dimensional Euclidean space is called* d*-configuration, if* $\dim(\mathcal{C}) = d$.

Definition 5. *Consider all geometrical intersections of all lines $l \in L$ and affine subspaces S of all subconfigurations of the d-configuration, where $l \not\subseteq S$ and $l \cap S \neq \emptyset$. A d-configuration is called natural d-configuration if each intersection point is also a point of the d-configuration. The natural d-configuration is denoted by $\mathcal{C}^{nd}$.*

Let's look at some examples: the geometric realization of the configuration $(6_2, 4_3)$ is a natural 2-configuration in figure 4.



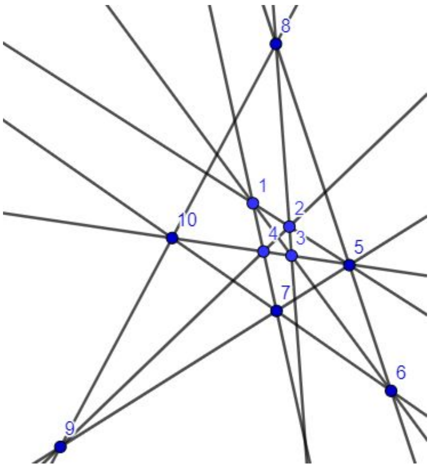
Geometric configuration $(6_2, 4_3)$



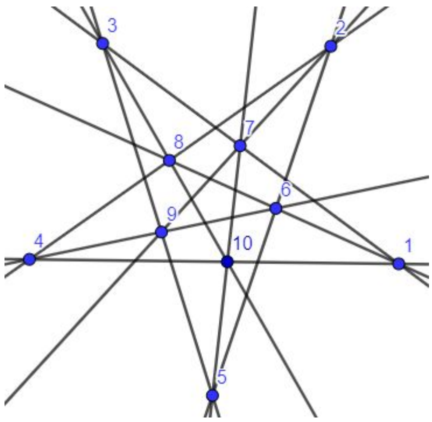
Geometric configuration (4_2)

Figure 4. Dimension of configurations

The geometric realization of the configuration (4_2) is not a natural 2-configuration, but its geometric realization in the 3-dimensional Euclidean space is a natural 3-configuration in figure 4. The geometric realization of the Desargues configuration $(10_3)_1$ is not a 2-configuration, but it is a natural 3-configuration in the 3-dimensional Euclidean space (figure 5).



Geometric configuration $(10_3)_1$



Geometric configuration $(10_3)_2$

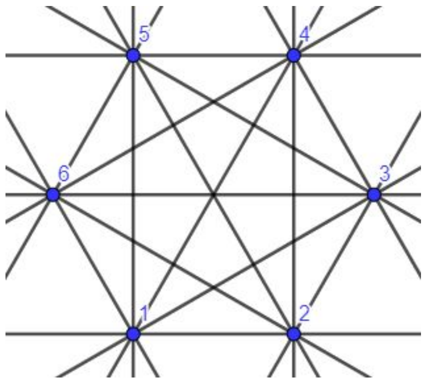
Figure 5. Dimension of configurations

The geometric realization of the non-Desargues configuration $(10_3)_2$ is a 2-configuration, but it is not a natural 2-configuration (figure 5).

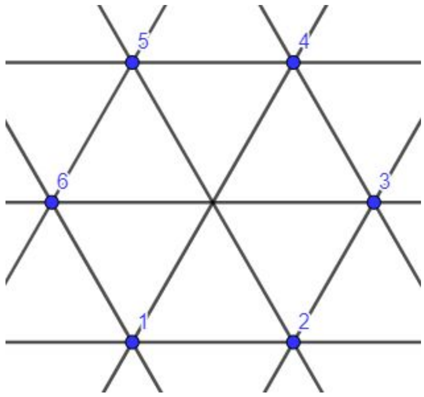
3 Theorems

Theorem 1. *The dimension of the configuration $\mathcal{C}(S) = \left(p_{p-1}, \binom{p}{2}_2\right)$ is $p-1$. The configuration $\mathcal{C}(S)$ can be realized in the $(p-1)$-dimensional space and this $(p-1)$-configuration is natural. Namely the configuration $\left(p_{p-1}, \binom{p}{2}_2\right)$ is a natural $(p-1)$-configuration.*

Proof. Consider p points of the configuration $\mathcal{C}(S)$. Integer $p-1$ is the largest number for which affine hull of p points has dimension $p-1$. In this case, p points form a $(p-1)$-dimensional simplex. Side lines of this simplex are lines of the configuration $\mathcal{C}(S)$. It means that $\dim(\mathcal{C}(\mathcal{S})) = p-1$. This configuration is denoted by $\mathcal{C}^{p-1}(S)$. Any line l of the configuration is the side line of the simplex, and any subspace S is a face of the simplex, consequently the intersection $l \cap S$ is a point of the configuration if $l \not\subseteq S$ and $l \cap S \neq \emptyset$. Thus the configuration $\mathcal{C}(S)$ is a natural $(p-1)$-configuration. □



Geometric configuration $(6_5, 15_2)$



Geometric configuration $(6_3, 9_2)$

Figure 6. 5-configurations

Theorem 2. *The dimension of the configuration* (p_q, n_2) *is* $p-1$*, where* $2 \le q \le p-1$ *and* $p \le n \le \binom{p}{2}$*. The configuration* (p_q, n_2) *is a natural* $(p-1)$*-configuration.*

Proof. This follows from the previous theorem, since the points of the configuration (p_q, n_2) are the same as the points of the configuration $\mathcal{C}(S)$, and its lines are part of the lines of the configuration $\mathcal{C}(S)$. □

Theorem 3. *The dimension of the configuration* (n_2) *is* $n-1$*. The configuration* (n_2) *is a natural* $(n-1)$*-configuration.*

Proof. It follows from the previous theorem by $p = n$, $q = 2$. □

Based on the above theorems, it seems useful to introduce the following concept.

Definition 6. *The subconfiguration* (p_q, n_2) *of a configuration* $\mathcal{C}$*, where* $2 \le q \le p-1$ *and* $p \le n \le \binom{p}{2}$ *is called a simplicial subconfiguration of the configuration* $\mathcal{C}$*, if the intersection of lines of the subconfiguration* (p_q, n_2) *is not point of the configuration* $\mathcal{C}$ *except points of the subconfiguration.*

If p is the number of points in the simplicial subconfiguration (p_q, n_2), then the maximum of p is denoted by S_{max}. If certain other conditions are satisfied, we suspect that the dimension of a configuration $\mathcal{C}$ is less than S_{max}. This estimation is sharp for the Desargues configuration, i.e. $S_{max} = d+1$, but not in general. The natural d-configuration $\mathcal{C}$ is called compact, if $S_{max} = d+1$.

Let's take a non-sharp example. In figure 7 the maximal number of points in the simplicial subconfigurations of the Cremona-Richmond configuration $\mathcal{C}^g(15_3)$ is 6. Consider the subconfiguration $(6_3, 9_2)$ of points $\{4, 5, 10, 11, 12, 13\}$. The intersections of lines of this subconfiguration are no points of the Cremona-Richmond configuration, consequently this subconfiguration is simplicial and the number of points is 6. But the dimension of the Cremona-Richmond configuration is 4 by [6, 8].

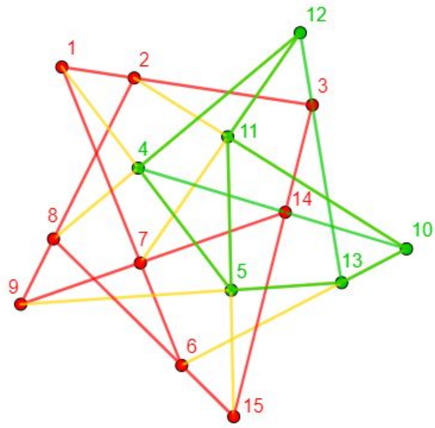


Figure 7. Cremona-Richmond configuration $\mathcal{C}^g(15_3)$

4 Summary

Grünbaum introduced the concept of the dimension of the configuration. We defined the natural d-configuration, where if the intersection of two lines of the configuration exists, then this point is also a point of the configuration in the d-dimensional Euclidean space. We proved that a few configurations in every d-dimensional Euclidean space are natural d-configurations and these configurations can also play an important role in determining the dimensions of other configurations.

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