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A STUDY OF THE NEIGHBORHOOD COMPLEX OF $<s,t>$-STABLE KNESER GRAPHS

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Abstract
In 1978, Alexander Schrijver defined the stable Kneser graphs as a vertex critical subgraphs of the Kneser graphs. In the early 2000s, Günter M. Ziegler generalized Schrijver's construction and defined the s-stable Kneser graphs. Thereafter Frédéric Meunier determined the chromatic number of the s-stable Kneser graphs for special cases and formulated a conjecture on the chromatic number of the s-stable Kneser graphs. In this paper we study a generalization of the s-stable Kneser graphs. For some specific values of the parameter we show that the neighborhood complex of $<s,t>$-stable Kneser graph has the same homotopy type as the $(t-1)$-sphere. In particular, this implies that the chromatic number of this graph is $t+1$.

1 Introduction

In this section, first we setup some notations and terminologies. Hereafter, the symbol $[m]$ stands for the set $\{1,\dots,m\}$. For positive integers $m \geq n$, let $2^{[m]}$ denote the collection of all subsets of $[m]$ and let $\binom{[m]}{n}$ denote the collection of all n-subsets of $[m]$.

For an integer $s \geq 2$ a subset $A \subseteq [m]$ is s-*stable* if any two of its elements are at least "at distance s apart" on the m-cycle, that is, if $s \leq |i-j| \leq m-s$ for distinct $i,j \in A$. A subset $A \subseteq [m]$ is *almost s-stable*, if $s \leq |i-j|$ for distinct $i,j \in S$. In this note we examine a further generalization of s-stable set. For positive integers $t \leq s$ a subset $A \subseteq [m]$ is $<s,t>$*-stable*, if

$$s \leq |i-j| \leq m-t$$

for distinct $i,j \in S$. The $<s,1>$-stable is the same as almost s-stable. There is a further generalization of stability as follows. For a n-set $A \subseteq [m]$, let $A(1), A(2), \dots, A(n)$ be the ordered elements of A, that is $A(1)$ is the smallest element and $A(n)$ is the largest element of A in the standard order. If $\vec{s} = (s_1,\dots,s_n)$ is an integer vector, then a n-subset $A \subseteq [m]$ is called $\vec{s}$*-stable*, if $s_j \leq A(j+1) - A(j)$ for $1 \leq j \leq n-1$ and $A(n) - A(1) \leq m - s_n$. Note that the cases $\vec{s} = (s,\dots,s)$, $\vec{s} = (s,\dots,s,1)$ and $\vec{s} = (s,\dots,s,t)$ demonstrate the usual concept of s-stable, almost s-stable and $<s,t>$-stable subsets, respectively.

Hereafter, the symbols $\binom{[m]}{n}_s$, $\binom{[m]}{n}_{s\sim}$, $\binom{[m]}{n}_{<s,t>}$, $\binom{[m]}{n}_{\vec{s}}$ stand for the collection of all s-stable, almost s-stable, $<s,t>$-stable and $\vec{s}$-stable n-subsets of $[m]$, respectively.

Let $\mathcal{F} \subseteq 2^{[m]}$ be a set system. The *Kneser graph of* $\mathcal{F}$ has $\mathcal{F}$ as the vertex set, and two sets $A, B \in \mathcal{F}$ are adjacent iff $A \cap B = \emptyset$. In the next section we will use the set systems $\binom{[m]}{n}_s$, $\binom{[m]}{n}_{s\sim}$, $\binom{[m]}{n}_{<s,t>}$, and $\binom{[m]}{n}_{\vec{s}}$ as the ground set of a Kneser graph.

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2 History and Motivation

2.1 On the chromatic number of the Kneser graphs

In 1978, László Lovász [11] proved the famous Kneser Conjecture, which state that the chromatic number of the Kneser graph $KG_{m,n}$ is equal to $m-2n+2$, where $KG_{m,n}$ denote the Kneser graph of $\binom{[m]}{n}$.

Theorem 1 (Lovász[11](1978)). *Let m,n be positive integers such that $m \geq 2n$. Then*

$$\chi(KG_{m,n}) = m - 2(n-1). \tag{1}$$

Shortly afterwards Alexander Schrijver [17] constructed a vertex critical subgraph $SG_{m,n}$ of $KG_{m,n}$. The stable Kneser graph $SG_{m,n}$ was obtained by restricting the vertex set to the n-subsets that are $2-stable$, that is the Kneser graph of $\binom{[m]}{n}_2$.

Theorem 2 (Schrijver[17](1978)). *Let m,n be positive integers such that $m \geq 2n$. Then*

$$\chi(SG_{m,n}) = m - 2(n-1). \tag{2}$$

Günter M. Ziegler generalized Schrijver's construction in [18] and gave an upper bound for the chromatic number of s-stable Kneser graph. The s-stable Kneser graph, denoted as $SG^s_{m,n}$, is the Kneser graph of $\binom{[m]}{n}_s$ for positive integers $n, s \geq 2$ and $m \geq sn$.

Proposition 3 (Ziegler[18](2002)). *Let m,n be positive integers such that $m \geq 2n$. Then*

$$\chi(SG^s_{m,n}) \leq m - s(n-1). \tag{3}$$

The almost s-stable Kneser graph, denoted as $SG^{s\sim}_{m,n}$, was defined by Frédéric Meunier in [12]. $SG^{s\sim}_{m,n}$ is the Kneser graph of $\binom{[m]}{n}_{s\sim}$ for positive integers $n, s \geq 2$ and $m \geq s(n-1)+2$. Meunier proved the following result about the chromatic number of almost 2-stable Kneser graph $SG^{2\sim}_{m,n}$.

Proposition 4 (Meunier[12](2011)). *Let m,n be positive integers such that $m \geq 2n$. Then*

$$\chi(SG^{2\sim}_{m,n}) = m - 2(n-1). \tag{4}$$

Furthermore, Meunier formulated the following conjecture about the chromatic number of s-stable Kneser graph $SG^s_{m,n}$.

Conjecture 5 (Meunier[12](2011)). *Let m,n,s be positive integers such that $m \geq sn$ and $s \geq 2$. Then*

$$\chi(SG^s_{m,n}) = m - s(n-1). \tag{5}$$

As noted above the conjecture is known to be true for $s = 2$ by the result of Schrijver [17]. In the case $m = sn$ the conjecture is trivially true, because $SG^s_{sn,n}$ is a complete graph. In addition Meunier [12] settled the case $m = sn + 1$. Further Jakob Jonsson [8] confirmed it for $s \geq 4$, provided m is sufficiently large in terms of s and n. Then in 2015, Peng-An Chen [4] proved the conjecture for even s.

Theorem 6 (Chen[4](2015)). *Let m,n,s be positive integers such that $m \geq sn$ and $s \geq 2$. If s is even, then*

$$\chi(SG^s_{m,n}) = m - s(n-1). \tag{6}$$

Afterwards Chen proved the following result about the chromatic number of almost s-stable Kneser graph $SG^{s\sim}_{m,n}$.

Theorem 7 (Chen[5]). *Let m,n,s be positive integers such that $m \geq sn$ and $s \geq 2$. Then*

$$\chi(SG^{s\sim}_{m,n}) = m - s(n-1). \tag{7}$$

Recently, in a joint work with Hamid Reza Daneshpajouh we proved the following theorem on the chromatic number of $\vec{s}$-stable Kneser graphs.

Theorem 8 (Daneshpajouh, Osztényi[6](2021)). *Let m,n be positive integers and $\vec{s}=(s_1,\ldots,s_n)$ be an integer vector where $n\geq 2$, $m\geq\sum_{i=1}^{n-1}s_i+2$, $s_i\geq 2$ for $i\neq n$ and $s_n\in\{1,2\}$. Then*

$$\chi\left(SG^{\vec{s}}_{m,n}\right)=m-\sum_{i=1}^{n-1}s_i. \tag{8}$$

In this note, in conjunction with Lovász's topological bound on the chromatic number, we can determine the chromatic number of $<s,t>$-stable Kneser graphs for some special parameters.

2.2 On the neighborhood complex of the Kneser graphs

In the proof of the Kneser conjecture Lovász introduced a simplicial complex related to a graph G, the neighborhood complex $\mathcal{N}(G)$ and applied the Borsuk–Ulam theorem to show that the connectivity of this complex give a lower bound for the chromatic number of the graph G.

Theorem 9 (Lovász[11](1978)). *If $\mathcal{N}(G)$ is $(k-2)$-connected, then*

$$\chi(G)>k. \tag{9}$$

Furthermore, he showed that the neighborhood complex of the Kneser graph $KG_{m,n}$ is $(m-2n-1)$-connected, which implies $\chi(KG_{m,n})>m-2n+1$. Schrijver used another, Bárány's method [1] to obtain the chromatic number of the stable Kneser graphs. Later, just in 2002 Anders Björner and Mark de Longueville [3] studied the neighborhood complex of $SG_{m,n}$, and showed that it has the homotopy type of the $(m-2n)$-sphere.

Theorem 10 (Björner, de Longueville[3](2002)). *For all positive integers m,n the complex $\mathcal{N}(KG_{m,n})$ is homotopy equivalent to the $(m-2n)$-sphere.*

Meunier, and the other authors of the papers devoted to the study of the chromatic number of the s-stable Kneser graphs [4, 5, 8], used combinatorial (Tucker–Ky Fan's lemma and $\mathbb{Z}_p$-Tucker lemma), or algebraic tools.

Recently, a result about the homotopy type of the neighborhood complex of almost s-stable Kneser graphs has been announced by the author of this note in [16].

Theorem 11 (Osztényi[16](2019)). *For all positive integers m,n,s such that $2\leq s$ and $s(n-1)+2\leq m$ the complex $\mathcal{N}(SG^{s\sim}_{m,n})$ is homotopy equivalent to the $(m-s(n-1)-2)$-sphere.*

Combining this result with Lovász's topological lower bound on the chromatic number of graphs yielded a new proof about the chromatic number of almost s-stable Kneser graphs $SG^{s\sim}_{m,n}$, which was determined earlier by Chen [5] in another method.

Also recently, Nandini Nilakanta and Anurag Singh [14, 15] constructed a maximal subgraph $S_{4+k,2}$ of $KG_{4+k,2}$ and $S_{6+k,3}$ of $KG_{6+k,3}$, whose neighborhood complex is homotopy equivalent to the neighborhood complex of $SG_{4+k,2}$ and $SG_{6+k,3}$, respectively. Further, they proved that the neighborhood complex of $S_{2n+k,n}$ deformation retracts onto the neighborhood complex of $SG_{2n+k,n}$ for $n=2,3$.

In the joint work with Hamid Reza Daneshpajouh we proved the following theorem on the neighborhood complex of $\vec{s}$-stable Kneser graphs.

Theorem 12 (Daneshpajouh, Osztényi[6](2021)). *Let m,n be positive integers and $\vec{s}=(s_1,\ldots,s_n)$ be an integer vector where $n\geq 2$, $m\geq\sum_{i=1}^{n-1}s_i+2$, $s_i\geq 2$ for $i\neq n$ and $s_n\in\{1,2\}$. Then, the neighborhood complex of $SG^{\vec{s}}_{m,n}$ is homotopy equivalent to the $\left(m-\sum_{i=1}^{n-1}s_i-2\right)$-sphere.*

The main purpose of this note is to present the homotopy type of the neighborhood complex of $<s,t>$-stable Kneser graphs for some special parameters.

3 Preliminaries

We now recall some basic definitions of combinatorial topology in order to fix notation, and raise some tools, which we will apply. The interested reader is referred to [2], [10] and [13] for more details.

A *simplicial complex* $\mathcal{K}$ is a set $V(\mathcal{K})$ (the *vertex set*) together with a hereditary set system of non-empty finite subsets of $V(\mathcal{K})$ (called *simplices*). We denote by Δ_n the simplicial complex of a n-dimensional simplex, that is $\Delta_n = 2^{[n+1]}$, and by $\dot{\Delta}_n$ its boundary complex. In this paper, whenever we make topological statements about a simplicial complex $\mathcal{K}$ we have the geometric realization $|\mathcal{K}|$ in mind.

A topological space X is k*-connected* if every map from the sphere $\mathbb{S}^n \to X$ extends to a map from the ball $\mathbb{B}^{n+1} \to X$ for $n = 0, 1, \ldots, k$. A space X is -1-connected if it is non-empty.

Two continuous maps $f, g : X \to Y$ are *homotopic* (written $f \sim g$) if there is a continuous map $F : X \times [0, 1] \to Y$ such that, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Two spaces X and Y are *homotopy equivalent* (or have the same homotopy type) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composition $f \circ g : Y \to Y$ is homotopic to the identity map id_Y and $g \circ f \sim id_X$. A space that is homotopy equivalent to a single point is called *contractible*.

For topological spaces X, Y, the *join* $X * Y$ is the quotient space $X \times Y \times [0, 1]/ \approx$, where the equivalence relation $\approx$ is given by $(x, y, 0) \approx (x', y, 0)$ for all $x, x' \in X$ and $y \in Y$ and $(x, y, 1) \approx (x, y', 1)$ for all $x \in X$ and $y, y' \in Y$. The *suspension of* X is the join with a two-point space: $susp(X) := X * \mathbb{S}^0$.

For a graph G, let $2^{V(G)}$ denote the set of all subsets of the vertex set of G. We denote by $cn \colon 2^{V(G)} \to 2^{V(G)}$ *the common neighborhood map*, introduced by Lovász [11]

$$cn(A) := \{v \in V(G) : (v, a) \in E(G) \text{ for all } a \in A\}.$$

The *neighborhood complex* $\mathcal{N}(G)$ is the simplicial complex whose vertices are the vertices of G and whose simplices are those subsets of $V(G)$ which have a common neighbor:

$$\mathcal{N}(G) = \{A \subseteq V(G) : \exists v \in V(G) \text{ that } A \subseteq cn(v)\}. \tag{10}$$

We will apply the idea of Björner and de Longueville [3] to deduce the homotopy equivalence of the complex $\mathcal{N}(SG^{<s,t>}_{m,n})$ and $\mathbb{S}^{t-1}$, that is we use the following consequence of the Gluing Lemma.

Proposition 13 ([3]). *Let $\mathcal{K}$ be a simplicial complex and $\mathcal{C}, \mathcal{D}$ contractible subcomplexes such that $\mathcal{K} = \mathcal{C} \cup \mathcal{D}$. Then $\mathcal{K}$ is homotopy equivalent to the suspension $susp(\mathcal{C} \cap \mathcal{D})$.*

4 On the neighborhood complex of the $<s, t>$-stable Kneser graphs

We have mentioned that the s-stable Kneser graph $SG^s_{sn,n}$ is a complete graph, and so the complex $\mathcal{N}(SG^s_{sn,n})$ is the boundary of the s-simplex, $\dot{\Delta}_s$. Similarly the $< s, t >$-stable Kneser graph $SG^{<s,t>}_{s(n-1)+t,n}$ is a complete graph with vertices $A_i = \{i, s + i, 2s + i, \ldots, (n - 1)s + i\}$ for $i = 1, \ldots, t$. So the complex $\mathcal{N}(SG^{<s,t>}_{s(n-1)+t,n})$ is the boundary of the t-simplex.

Next we study the neighborhood complex $\mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n})$.

Proposition 14. *For all positive integers n, s, t such that $t < s$ and $2 \leq n$ the complex $\mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n})$ is homotopy equivalent to the $(t - 1)$-sphere.*

Proof. The proof is by induction on t. The base case is known to be true for $t = 1$ by the result of [16].

Now, assume that $t > 1$. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be the subcomplexes of $\mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n})$ defined by

$$\mathcal{N}_1 := \{F \subseteq cn(A) : A \in V(SG^{<s,t>}_{s(n-1)+t+1,n}) \text{ such that } 1 \in A\}, \tag{11}$$

$$\mathcal{N}_2 := \{F \subseteq cn(B) : B \in V(SG^{<s,t>}_{s(n-1)+t+1,n}) \text{ such that } 1 \notin B\}, \tag{12}$$

Clearly $\mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n}) = \mathcal{N}_1 \cup \mathcal{N}_2$. Now, Theorem 13 ensures, that it suffices to show that $\mathcal{N}_1$ and $\mathcal{N}_2$ are contractible and $\mathcal{N}_1 \cap \mathcal{N}_2$ is homotopy equivalent to the $(t-2)$-sphere.

First we deduce that $\mathcal{N}_1$ and $\mathcal{N}_2$ are contractible. For this we define the following $< s, t >$-stable n-subsets of $[s(n-1)+t+1]$

$$A_{i,0} := \{i, s+i, 2s+i, \ldots, (n-1)s+i\} \tag{13}$$

for $1 \le i \le t+1$, and

$$A_{i,j} := \{i, s+i, 2s+i, \ldots, (n-j-1)s+i, (n-j)s+i+1, (n-j+1)s+i+1, \ldots, (n-1)s+i+1\} \tag{14}$$

for $1 \le j \le n-1$ and $1 \le i \le t$.

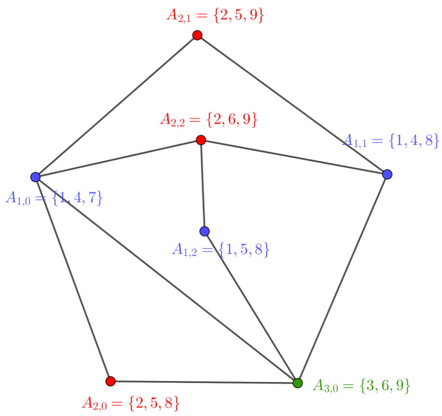

Figure 1. The Kneser graph $SG^{<3,2>}_{9,3}$.

It is easy to see that

$$V(SG^{<s,t>}_{s(n-1)+t+1,n}) = \{A_{i,0} : \text{ for } 1 \le i \le t+1\} \cup \{A_{i,j} : \text{ for } 1 \le j \le n-1,\ 1 \le i \le t\}. \tag{15}$$

Furthermore

$$\mathcal{N}_1 = \{F \subseteq cn(A_{1,j}) : \text{ for } 0 \le j \le n-1\}, \tag{16}$$

$$\mathcal{N}_2 = \{F \subseteq cn(A_{i,0}) : \text{ for } 2 \le i \le t+1\} \cup \{F \subseteq cn(A_{i,j}) : \text{ for } 1 \le j \le n-1,\ 2 \le i \le t\} \tag{17}$$

The subset $A_{1,0}$ is disjoint from all $A_{i,0}$ for $2 \le i \le t+1$ and from $A_{i,j}$ for $1 \le j \le n-1,\ 2 \le i \le t$, thus

$$\mathcal{N}_1 = \{F \subseteq cn(A_{1,0})\} \tag{18}$$

is a simplex. In addition to this $A_{1,0} \in cn(A_{i,0})$ for $2 \le i \le t+1$ and $A_{1,0} \in cn(A_{i,j})$ for $1 \le j \le n-1,\ 2 \le i \le t$, that is $\mathcal{N}_2$ is a cone. Thus we have shown that $\mathcal{N}_1, \mathcal{N}_2$ is contractible

To deduce that the complex $\mathcal{N}_1 \cap \mathcal{N}_2$ is homotopy equivalent to the $(t-2)$-sphere we will prove that $\mathcal{N}_1 \cap \mathcal{N}_2$ is isomorphic to $\mathcal{N}(SG^{<s,t-1>}_{s(n-1)+t,n})$. Notice that

$$V(SG^{<s,t-1>}_{s(n-1)+t,n}) = \{A_{i,0} : \text{ for } 1 \le i \le t\} \cup \{A_{i,j} : \text{ for } 1 \le j \le n-1,\ 1 \le i \le t-1\}. \tag{19}$$

The following graph homomorphism $\phi : SG^{<s,t-1>}_{s(n-1)+t,n} \to SG^{<s,t>}_{s(n-1)+t+1,n}$

$$\phi(A_{i,j}) = A_{i+1,j}, \text{ for all possible values of the pair } i, j, \tag{20}$$

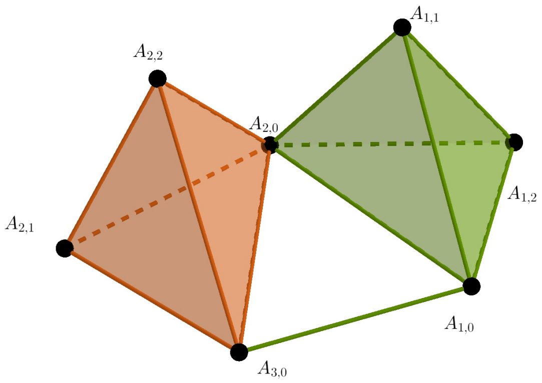

Figure 2. The neighborhood complex of the Kneser graph $SG_{9,3^{<3,2>}}$.

induces a simplicial mapping

$$\Phi : \mathcal{N}(SG^{<s,t-1>}_{s(n-1)+t,n}) \to \mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n}). \tag{21}$$

For this simplicial mapping we have

$$\Phi(cn(A_{i,j})) = cn(A_{1,0}) \cap cn(\phi(A_{i,j})), \tag{22}$$

so

$$\Phi : \mathcal{N}(SG^{<s,t-1>}_{s(n-1)+t,n}) \to \mathcal{N}_1 \cap \mathcal{N}_2 \tag{23}$$

is an isomorphism.

So we completed the proof □

				$n = 3$					
$<s,t>$	m	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
<4,2>	10	$\mathbb{Z}$							
	11		$\mathbb{Z}$						
	12			$\mathbb{Z}$					
<4,3>	11		$\mathbb{Z}$						
	12			$\mathbb{Z}$					
	13				$\mathbb{Z}^3$				
<4,4>	12			$\mathbb{Z}$					
	13				$\mathbb{Z}$				
	14				$\mathbb{Z}$	$\mathbb{Z}^2$			

Table 1. The reduced integer homology groups of $\mathcal{N}(SG^{<4,t>}_{m,3})$ in some cases.

Remark. With the help of the software `polymake`, we have computed the reduced integer homology groups of $\mathcal{N}(SG^{<s,t>}_{m,n})$ in some cases (see Table 1. and Table 2.). For the Kneser graphs $SG^{<5,2>}_{m,2}$ and $SG^{<4,2>}_{m,3}$, we obtained the results already known in [3]. For the cases $2 < t < s$ these results showed that the connectivity of $\mathcal{N}(SG^{<s,t>}_{m,n})$ is larger than $m - s(n-1) - t - 1$, but these complexes are not homotopy equivalent to a sphere in general. In additionally, we studied the case $t = s$ also, and we obtained the results already known in [16]. That is the neighborhood complexes of these graphs are not homotopy equivalent to a sphere, in addition the connectivity of the neighborhood complexes of these graphs does not give the expected lower bound for the chromatic number of these graphs.

				$n=2$					
$<s,t>$	m	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
<5,2>	7	$\mathbb{Z}$							
	8		$\mathbb{Z}$						
	9			$\mathbb{Z}$					
	10				$\mathbb{Z}$				
<5,3>	8		$\mathbb{Z}$						
	9			$\mathbb{Z}$					
	10				$\mathbb{Z}$				
	11					$\mathbb{Z}$	$\mathbb{Z}^2$		
<5,4>	9			$\mathbb{Z}$					
	10				$\mathbb{Z}$				
	11					$\mathbb{Z}$			
	12						$\mathbb{Z}^2$	$\mathbb{Z}$	
<5,5>	10			$\mathbb{Z}$					
	11				$\mathbb{Z}$				
	12					$\mathbb{Z}^6$	$\mathbb{Z}$		
	13						$\mathbb{Z}_2$		$\mathbb{Z}$

Table 2. The reduced integer homology groups of $\mathcal{N}(SG^{<5,t>}_{m,2})$ in some cases.

5 On the chromatic number of the $<s,t>$-stable Kneser graphs

The s-stable Kneser graph $SG^{s}_{sn,n}$ is a complete graph, thus we can colour it with s color. Meunier proved in [12] that the chromatic number of $SG^{s}_{sn+1,n}$ is equal with $s+1$. Now we give the generalization of this result for $<s,t>$-stable Kneser graph in $t < s$ cases.

Theorem 15. *Let n, s, t be positive integers such that $s \geq 2$ and $t < s$. Then*

$$\chi(SG^{<s,t>}_{s(n-1)+t+1,n}) = t+1. \tag{24}$$

Proof. The topological connectivity of the sphere $\mathbb{S}^{t-1}$ is $t-2$, consequently Lovász's topological lower bound gives the following lower bound for the chromatic number:

$$\chi(SG^{s,t}_{s(n-1)+t+1,n}) > conn(\mathcal{N}(SG^{<s,t>}_{s(n-1)+t+1,n})) + 2 = t. \tag{25}$$

Furthermore, it is easy to see that the usual coloring uses at most $t+1$ colors, and is proper: for a vertex A of $SG^{<s,t>}_{s(n-1)+t+1,n}$, we define its colors by

$$c(A) := \min A. \tag{26}$$

That is the chromatic number of $SG^{<s,t>}_{s(n-1)+t+1,n}$ is equal to $t+1$. □

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