ON EXTRACTION OF P-CONFIGURATIONS

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Abstract
Consider a parallelotope \( P \) and its dual polytope \( P^* \). The parallelotope configuration or the p-configuration is a system of lines and points projected vertices and edges of the polytope \( P^* \) from the center of the polytope \( P^* \) to a special \((n−1)\)-dimensional hyperplane. The extraction of the parallelotope defines an extraction of the p-configuration, as well. In this paper we examine properties of the extraction of the p-configuration.

1 Configurations

I have described the properties of the parallelotopes and of the p-configurations in details in [7]. In the current paper I will briefly summarize the most important concepts and I will investigate the extraction of the p-configurations.

A configuration is a system of \( p \) points and \( g \) straight lines arranged in a plane in such a way that every point of the system is incident with a fixed number \( γ \) of straight lines and every straight line of the system is incident with a fixed number \( π \) of points. Notation: \((p,γ,\pi)\).

The following relation must be true for every configuration:

\[ p \cdot γ = g \cdot π. \]

The configurations in which the number of points is equal to the number of lines, i.e. for which \( p = g \) and consequently \( γ = π \) are called symmetric or balanced configurations. For such a configuration the notation \( p_γ \) is used by [3], [6].

\[ (3_2) \text{ symmetric} \quad (6_2, 4_3) \text{ nonsymmetric} \]

Figure 1. Configurations in the plane

2 Parallelotopes

The parallelotope \( P \) is a convex polytope whose translated copies tile the space in a face to face manner. The centers of the parallelotopes form an \( n \)-dimensional lattice. In the plane there are two types of parallelotopes: parallelograms and centrally symmetric hexagons.

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Parallelotopes were characterized by B. A. Venkov[8] and later P. McMullen[5] in the following way: a polytope \( P \) is a parallelotope if and only if \( P \) is centrally symmetric, each facet of \( P \) is centrally symmetric, and the 2-dimensional orthogonal projection along any \((n - 2)\)-face of \( P \) is either a parallelogram or a centrally symmetric hexagon.

The edges of the parallelogram and the centrally symmetric hexagon are projections of the facets of the parallelotope \( P \). These facets form a 4- or a 6-belt, respectively.

B. A. Venkov introduced the concept of the parallelotope of non-zero width in the direction of a \( k \)-subspace \( X^k \). A parallelotope \( P \) has non-zero width along \( X^k \) if the intersection \( P \cap (X^k + a) \) is either \( k \)-dimensional or empty for every translation vector \( a \). Denote by \( F^{n-1} \) a facet of the parallelotope \( P \) and by \( t \) the lattice vector between the centers of the two nearest parallelotopes \( P \) and \( Q \) where \( P \) and \( Q \) have the common facet \( F^{n-1} \). This lattice vector is called relevant vector of the facet \( F^{n-1} \) by [2].

**Theorem 1.** (B. A. Venkov [9]) Let \( P \) be an \( n \)-dimensional parallelotope of non-zero width along \( X^k \). Then the projection of \( P \) along \( X^k \) is a parallelotope (of dimension \( n - k \)) and the lattice vectors \( t \), related to the facets \( F^{n-1} \) which are parallel to \( X^k \) generate an \((n - k)\)-dimensional lattice spanning a space \( X^{n-k} \) which is transversal to \( X^k \).

By this theorem for \( k = n - 2 \) relevant vectors of a 4- or 6-belt span a 2-dimensional lattice, thus these relevant vectors are in a plane.

Consider the parallelotopes \( P \) and \( Q \) of dimension \( n \). For \( k = 1 \) denote by \( S(z) \) the segment \( X^k \) of the direction \( z \) and of the length \( z \). If there exists a direction \( z \) for which \( P \oplus S(z) = Q \), where \( \oplus \) denotes the Minkowski sum, then \( P \) is called the contraction of \( Q \) and \( Q \) is the extraction of \( P \). The following theorem provides an important condition for the extraction of the parallelotope \( P \) to be a parallelotope.

**Theorem 2.** [1] Let \( P \) be a parallelotope and \( z \) be a vector. \( P \oplus S(z) \) is a parallelotope if and only if \( z \) is parallel to at least one facet of each 6-belt.

**Definition 1.** The shadow boundary of a parallelotope \( P \) in the direction \( z \) consists of all boundary points \( x \) of \( P \) for which the line \( \{ x + \lambda z | \lambda \in \mathbb{R} \} \) is a support line of \( P \). (There is no point of the line \( \{ x + \lambda z | \lambda \in \mathbb{R} \} \) belonging to the interior of \( P \)). It is denoted by \( sh_z(P) \).

It is well known that the shadow boundary of a convex polytope is the union of its several \((n - 1)\)- and \((n - 2)\)-dimensional closed faces [4].

### 3 The p-configurations

Every \( n \)-polytope has a dual structure. It can be obtained by interchanging its vertices and facets, edges and \((n - 2)\)-dimensional faces, and so on, generally interchanging its \((j - 1)\)-dimensional elements and \((n - j)\)-dimensional elements, preserving coincidence between elements. We use duality for parallelotopes. The dual polytope of the parallelotope \( P \) is denoted by \( P^* \). Facets of a parallelotope \( P \) are centrally symmetric, thus we can define vertices of dual polytope \( P^* \) by centers of facets of the parallelotope \( P \). For example consider the blue truncated octahedron in the picture 4. The dual polytope is the red polytope, which is not a parallelotope.
Consider the dual polytope $P^*$ of a parallelotope $P$. The parallelotope configuration or the p-configuration is a system of lines and points projected vertices and edges of the polytope $P^*$ from the center of the polytope $P^*$ to an $(n-1)$-dimensional hyperplane, which is parallel to a hyperplane containing the center of the parallelotope $P$ and not containing any vertex of the dual polytope $P^*$. In this manner every parallelotope $P$ determines a p-configuration denoted by $\Pi$. A line of a p-configuration is called a p-line. A vertex of a p-configuration is called a p-vertex.

The notation of the p-configuration is

$$((p_1)_{\gamma_1}, (p_2)_{\gamma_2}, \ldots, (p_k)_{\gamma_k}; ((g_1)_{\pi_1}, (g_2)_{\pi_2}, \ldots, (g_l)_{\pi_l}),$$

where $p_i$ is the number of points which belong to $\gamma_i$ lines and $g_i$ is the number of lines which contain $\pi_i$ points. The sum of $p_i$ is equal to the number of all points of the p-configuration and the sum of $g_i$ is equal to the number of all lines of the p-configuration.

I proved the following properties of the p-configuration in [7].

**Theorem 3.** Every p-line contains two or three points of the p-configuration $\Pi$. Every point of the p-configuration $\Pi$ belongs to at least $(n-1)$ straight lines, these lines generate a $(n-1)$-dimensional hyperplane.

**Theorem 4.** The intersection of two uniplanar p-line is a point of the p-configuration $\Pi$.

**Definition 2.** The p-configurations $\Pi_1$ and $\Pi_2$ are combinatorially equivalent, if there is a bijection between the p-vertices and the p-lines of the p-configurations $\Pi_1$ and $\Pi_2$ which preserves the point-line coincidence. We denote combinatorially equivalent p-configurations by $\Pi_1 \cong \Pi_2$.

4 Paralleloptopes and p-configurations

The pair of two centrally symmetric $k$-dimensional faces of the parallelotope $P$ is denoted by $P F^k$ and the set of parallel $k$-dimensional faces of the parallelotope $P$ is denoted by $\langle P F^k \rangle$. Similarly the pair of two centrally symmetric $k$-dimensional faces of the dual polytope $P^*$ of the parallelotope $P$ is denoted by $P^* F^k$. If it is not ambiguous, the two opposite $k$-dimensional faces $P F^k$ are called $k$-dimensional face. Similarly two opposite vertices $P F^0$ are called vertex of the parallelotope $P$.

Consider a $k$-dimensional convex polytope. If all vertices and all edges of the polytope are p-vertices and p-segments of the p-configuration and the interior of the polytope does not contain p-vertices or p-segments, then the system of p-vertices and p-segments of the polytope is called $k$-dimensional face of the p-configuration and it is denoted by $\Pi F^k$. We remark that the convex polytope is defined in projective sense. The $k$-dimensional subspace containing the face $\Pi F^k$ is denoted by $\langle \Pi F^k \rangle$ and the $(n-1)$-dimensional hyperplane containing the p-configuration $\Pi$ is denoted by $\langle \Pi \rangle$. 

![Figure 2. Parallelotope configuration](image-url)
Lemma 1. Consider an \((n-1)\)-dimensional p-configuration II corresponding to an \(n\)-dimensional parallelotope \(P\). There is a bijection between an \((n-k)\)-dimensional face \(PF_{n-k}\) of the parallelotope \(P\) and a \((k-1)\)-dimensional face \(\Pi F_{k-1}\) of the p-configuration II.

**Proof.** By the duality there is a bijection between the \((n-k)\)-dimensional face \(PF_{n-k}\) of the parallelotope \(P\) and the \((k-1)\)-dimensional face \(P^*F_{k-1}\) of the dual polytope \(P^*\). The central projection maps a \((k-1)\)-dimensional face \(P^*F_{k-1}\) of the dual polytope \(P^*\) to a \((k-1)\)-dimensional polytope \(F^k\). The central projection preserves the convexity (in projective sense) and the coincidence. There is a one-to-one correspondence between the vertices and edges of the \((k-1)\)-dimensional face \(P^*F_{k-1}\) of \(P^*\) and the vertices and edges of the \((k-1)\)-dimensional polytope \(F^k\) and of II. Thus the \((k-1)\)-dimensional polytope \(F^k\) is a \((k-1)\)-dimensional face \(\Pi F_{k-1}\) of II.

In other direction if \(\Pi F_{k-1}\) is a \((k-1)\)-dimensional face of II then vertices of \(\Pi F_{k-1}\) are the central projections of vertices of the dual polytope \(P^*\). The interior of the convex hull \(C\) of these vertices contains only boundary points of \(P^*\) otherwise the interior of \(\Pi F_{k-1}\) contains p-vertices or p-segments of II, as well. Thus the convex hull \(C\) is a \((k-1)\)-dimensional face of II because if the dimension of the convex hull \(C\) is not \((k-1)\), then the dimension of the face of II is different, as well. A \((n-k)\)-dimensional face of the parallelotope \(P\) corresponds to this \((k-1)\)-dimensional face of II. Consequently there is a bijection between the \((n-k)\)-dimensional face \(PF_{n-k}\) and the \((k-1)\)-dimensional face \(\Pi F_{k-1}\).

Further on \(\Pi F_{k-1}\) and \(PF_{n-k}\) denote the corresponding faces of the p-configuration II and of the parallelotope \(P\).

Lemma 2. Consider an \((n-k)\)-dimensional face \(PF_{n-k}\) and an \((n-j)\)-dimensional face \(PF_{n-j}\) of the parallelotope \(P\) and the corresponding \((k-1)\)-dimensional face \(\Pi F_{k-1}\) and \((j-1)\)-dimensional face \(\Pi F_{j-1}\) of the \((n-1)\)-dimensional p-configuration II. \(PF_{n-j}\) is contained in \(PF_{n-k}\) if and only if \(\Pi F_{k-1}\) is contained in \(\Pi F_{j-1}\).

**Proof.** By the property of the duality \(PF_{n-j}\) is contained in \(PF_{n-k}\) if and only if \(P^*F_{k-1}\) is contained in \(P^*F_{j-1}\), where \(P^*F_{k-1}\) and \(P^*F_{j-1}\) are faces of the dual polytope \(P^*\). The central projection preserves the relation \(\subseteq\), consequently \(P^*F_{k-1}\) is contained in \(P^*F_{j-1}\) if and only if \(\Pi F_{k-1}\) is contained in \(\Pi F_{j-1}\).

Lemma 3. Consider an \((n-j)\)-dimensional face \(PF_{n-j}\) and all \((n-k)\)-dimensional faces \(PF_i^{n-k}\) of the parallelotope \(P\) for which \(PF_{n-j}\) is contained in \(PF_i^{n-k}\) and consider the corresponding \((j-1)\)-dimensional face \(\Pi F_{j-1}\) and \((k-1)\)-dimensional faces \(\Pi F_{k-1}\) of the \((n-1)\)-dimensional p-configuration II. \(PF_{n-j}\) is contained in \(PF_i^{n-k}\) if and only if \(\Pi F_{j-1}\) is contained in \(\Pi F_{k-1}\).

**Proof.** Similarly to lemma 2 it is easy to see that the statement is satisfied by the property of the duality and the central projection. We remark that the faces \(\Pi F_{k-1}\) are all \((k-1)\)-dimensional faces of the \((j-1)\)-dimensional face \(\Pi F_{j-1}\) for which \(\Pi F_{k-1}\) is contained in \(\Pi F_{j-1}\). Q.e.d.

By the definition of the p-configuration it can easily be seen that there is a bijection between a point \(\langle \Pi F^0 \rangle\) of II and a facet \(\langle PF_{n-1}^0 \rangle\) of \(P\) and between a segment \(\Pi F^1\) of II and an \((n-2)\)-dimensional face \(PF_{n-2}\) of \(P\). Similarly according to theorem 3 a 1-dimensional line \(\langle \Pi F^1 \rangle\) corresponds in a one-to-one manner to parallel \((n-2)\)-dimensional faces \(\langle PF_{n-2} \rangle\).

By lemma 1 there is a bijection an \((n-1)\)-dimensional face \(\Pi F_{n-1}\) of II and the vertex \(PF^0\) of \(P\). The set \(\langle PF^0 \rangle\) consists of all vertices of the parallelotope \(P\), and the set of all facets \(P^*F_{n-1}\) of the dual polytope \(P^*\) corresponds to the set \(\langle PF^0 \rangle\). The central projection of all facets \(P^*F_{n-1}\) of the dual polytope \(P^*\) fills the \((n-1)\)-dimensional hyperplane without gaps. Consequently there is a one-to-one correspondence between the \((n-1)\)-dimensional hyperplane \(\langle \Pi F_{n-1} \rangle\) of II and vertices \(\langle PF^0 \rangle\) of \(P\) and therefore \(\langle \Pi F_{n-1} \rangle = \langle \Pi \rangle\).

These special cases are generalized in the following lemma.

Lemma 4. Let \(P\) be an \(n\)-dimensional parallelotope of non-zero width along \(X^{n-k}\) and let the elements of the set \(\langle PF_{n-k} \rangle\) be parallel to \(X^{n-k}\). There is a bijection between the set \(\langle PF_{n-k} \rangle\) of the parallelotope \(P\) and the set of a \((k-1)\)-dimensional subspace \(\langle \Pi F_{k-1} \rangle\) of the p-configuration II for which \(\langle \Pi F_{k-1} \rangle \cap \Pi \) is a \((k-1)\)-dimensional p-configuration \(\Pi_{k-1}\).
Proof.

Generally by theorem 1 the lattice vectors $t_i$ related to the facets $PF_i^{n-1}$ which are parallel to $X^{n-k}$ generate a $k$-dimensional lattice spanning a space $X^k$, where $i \in I$. The center points of these facets $PF_i^{n-1}$ generate the dual polytope $D^k$ of a $k$-dimensional parallelotope $\mathcal{P}^k$. By the definition of the p-configuration the central projection of the polytope $D^k$ is a $(k-1)$-dimensional p-configuration $\Pi^{k-1}$ and $\Pi^{k-1} \subseteq \Pi$.

By theorem 1 the space $X^k$ does not contain the center points of the facets which are not parallel to $X^{n-k}$, thus the $(k-1)$-dimensional hyperplane $\langle \Pi^{k-1} \rangle$ does not contain other p-line or p-vertex of the p-configuration $\Pi$, consequently

$$\langle \Pi^{k-1} \rangle \cap \Pi = \Pi^{k-1}. \quad (3)$$

Consider an arbitrary $(n-k)$-dimensional face $PF^{n-k} \in \langle PF^{n-k} \rangle$. By lemma 1, a $(k-1)$-dimensional face $\Pi F^{k-1}$ of $\Pi$ corresponds to an $(n-k)$-dimensional face $PF^{n-k}$ of $\mathcal{P}$. If $PF^{n-k} \subseteq PF^{n-1}_j$ for the facets $PF^{n-1}_j$ where $j \in J \subseteq I$, then $\Pi F^{0}_{j} \subseteq \Pi F^{k-1}$. On the other hand every vertex $\Pi F^{0}_{j} \subseteq \Pi F^{k-1}$, therefore $\Pi F^{k-1} \subseteq \Pi^{k-1}$, consequently $\langle \Pi F^{k-1} \rangle = \langle \Pi^{k-1} \rangle$ because both sides are $(k-1)$-dimensional hyperplanes. Thus

$$\langle \Pi F^{k-1} \rangle \cap \Pi = \langle \Pi^{k-1} \rangle \cap \Pi. \quad (4)$$

Consequently $\langle \Pi F^{k-1} \rangle \cap \Pi$ is a $(k-1)$-dimensional p-configuration $\Pi^{k-1}$.

Let $\Pi F^{k-1}$ be an arbitrary $(k-1)$-dimensional face of $\Pi$. If $\Pi F^{k-1} \subseteq \langle \Pi F^{k-1} \rangle$ and $\langle \Pi F^{k-1} \rangle \cap \Pi = \Pi^{k-1}$, then $\Pi F^{k-1} \subseteq \Pi^{k-1}$ and the facets $PF^{n-k}$ corresponding to all vertices $\Pi F^{0}_{j}$ of $\Pi F^{k-1}$ are parallel to $X^{n-k}$. According to lemma 3 $PF^{n-k} = \bigcap PF^{n-1}_i$ for all $i$, thus $PF^{n-k}$ is parallel to $X^{n-k}$, consequently $\Pi F^{n-k} \subseteq \langle PF^{n-k} \rangle$. Therefore there is a bijection between the set $\langle PF^{n-k} \rangle$ parallel to $X^{n-k}$ and the $(k-1)$-dimensional hyperplane $\langle \Pi F^{k-1} \rangle$ for which $\langle \Pi F^{k-1} \rangle \cap \Pi = \Pi^{k-1}$. Q.e.d.

5 Extraction of p-configuration

Let $\Pi$ be a p-configuration corresponding to a $n$-dimensional parallelotope $\mathcal{P}$. The dimension of $\Pi$ is $m = n - 1$.

Definition 3. If the parallelotope $\mathcal{P}$ has zero width in direction $z$ and $\mathcal{P} \oplus S(z)$ is an extraction of this parallelotope $\mathcal{P}$, then $\mathcal{P}$ and $\mathcal{P} \oplus S(z)$ determine two p-configurations denoted by $\Pi$ and $\Pi \oplus H(z)$. The p-configuration $\Pi \oplus H(z)$ is called an extraction of the p-configuration $\Pi$.

Theorem 5. Let $\mathcal{P}$ be a parallelotope and $z$ be a direction, such that $\mathcal{P}$ has zero width in direction $z$. Then $\Pi \oplus H(z) = \Pi \cup \Pi^{m-1}$, where $\Pi^{m-1}$ is an $(m-1)$-dimensional p-configuration.

Proof. For the segment $S(z)$ and an $n$-dimensional parallelotope $\mathcal{P}$ which has zero width in direction $z$ the parallelotope $\mathcal{P} \oplus S(z)$ contains 1-dimensional edges $\langle (\mathcal{P} \oplus S(z))F^1 \rangle$ parallel to $S(z)$. Using lemma 4 one can see that an $(m-1)$-dimensional hyperplane $\langle (\Pi \oplus H(z))F^{m-1} \rangle$ corresponds to parallel 1-dimensional edges $\langle (\mathcal{P} \oplus S(z))F^1 \rangle$ where $\langle (\Pi \oplus H(z))F^{m-1} \rangle \cap (\Pi \oplus H(z))$ is an $(m-1)$-dimensional p-configuration $\Pi^{m-1}$. Thus $\Pi^{m-1} \subseteq \Pi \oplus H(z)$ and $\Pi \subseteq \Pi \oplus H(z)$ consequently $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(z)$.

On the other hand consider an arbitrary $(k-1)$-dimensional face $\langle (\Pi \oplus H(z))F^{k-1} \rangle$ of the p-configuration $\Pi \oplus H(z)$. The $(k-1)$-dimensional face $\langle (\Pi \oplus H(z))F^{k-1} \rangle$ corresponds to the $(n-k)$-dimensional face $\langle \mathcal{P} \oplus S(z) \rangle F^{n-k}$ of the parallelotope $\mathcal{P} \oplus S(z)$. If the $(n-k)$-dimensional face $\langle \mathcal{P} \oplus S(z) \rangle F^{n-k}$ is parallel to the direction $z$ then $\langle (\Pi \oplus H(z))F^{k-1} \rangle \subseteq \Pi^{m-1}$. If the $(n-k)$-dimensional faces $\langle \mathcal{P} \oplus S(z) \rangle F^{n-k}$ is not parallel to the direction $z$ then $\langle (\Pi \oplus H(z))F^{k-1} \rangle \subseteq \Pi$. Consequently for an arbitrary $(k-1)$-dimensional face $\langle (\Pi \oplus H(z))F^{k-1} \rangle \subseteq \Pi \cup \Pi^{m-1}$. Therefore $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(z)$. By $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(z)$ and $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(z)$ the equality holds.
The above \((m - 1)\)-dimensional \(p\)-configuration \(\Pi^{m-1}\) is called a \(p\)-subconfiguration.

**Definition 4.** The \(p\)-subconfigurations \(\Pi_i^{m-1}\) and \(\Pi_2^{m-1}\) are combinatorially equivalent, if \(\Pi \cup \Pi_i^{m-1} \cong \Pi \cup \Pi_2^{m-1}\).

**Theorem 6.** Let \(\Pi \oplus H(z)\) be an extraction of the \(p\)-configuration \(\Pi\), then the intersection of the \(p\)-subconfiguration \(\Pi^{m-1}\) and an arbitrary \(p\)-line of the \(p\)-configuration \(\Pi\) is

\[(P1)\] a \(p\)-line of the \(p\)-configuration \(\Pi\)

\[(P2)\] a \(p\)-vertex of the \(p\)-configuration \(\Pi\),

\[(P3)\] a \(p\)-vertex of the \(p\)-configuration \(\Pi \oplus H(z)\), which is not a \(p\)-vertex of the \(p\)-configuration \(\Pi\). In this situation the \(p\)-line has to contain two \(p\)-vertices of \(\Pi\).

**Proof.** Case 1. If the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) are parallel to the direction \(z\), then the corresponding 1-dimensional \(p\)-line \(\langle II F^1 \rangle\) is a \(p\)-line of the \(p\)-configuration \(\Pi\) and of the \(p\)-subconfiguration \(\Pi^{m-1}\), as well.

Case 2. If the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) are not parallel to the direction \(z\), then the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) intersect the shadow boundary \(sh_z(\mathcal{P})\) in two opposite facets \(PF^{n-1}_i\) or two \((n - 2)\)-dimensional faces \(PF^{n-2}_i\) because the shadow boundary of a convex polytope is the union of its several \((n - 1)\)- and \((n - 2)\)-dimensional closed faces. If the intersection is two facets \(PF^{n-1}_i\), then the corresponding 1-dimensional \(p\)-line \(\langle II F^1 \rangle\) contains the corresponding \(p\)-vertex \(\langle II F^0 \rangle\).

Case 3. If the intersection is two \((n - 2)\)-dimensional faces \(PF^{n-2}_i\), then \(PF^{n-2}_i \oplus S(z)\) are facets \(PF^{n-1}_i\) of the parallelootope \(\mathcal{P} \oplus S(z)\), consequently a 1-dimensional \(p\)-line \(\langle II F^1 \rangle\) corresponding to the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) intersects the \(p\)-subconfiguration \(\Pi^{m-1}\) in a new \(p\)-vertex corresponding to two opposite facets \(\langle \mathcal{P} \oplus S(z) \rangle F^{n-1}_i\) of \(\langle \mathcal{P} \oplus S(z) \rangle\). If the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) of the parallelootope \(\mathcal{P}\) determine a 3-belt, then after the extraction it will be a 4-belt. Consequently the \((n - 2)\)-dimensional faces \(\langle PF^{n-2}_i \rangle\) of the parallelootope \(\mathcal{P}\) determine a 2-belt, thus the 1-dimensional \(p\)-line \(\langle II F^1 \rangle\) contains two \(p\)-vertices of the \(p\)-configuration \(\Pi\).
Definition 5. A p-line of the p-configuration \( \Pi \) is called a p3-line if the p-line contains three p-vertices. A p-vertex of the p-configuration \( \Pi \) is called a p3-vertex, if the p-vertex belongs to at least one p3-line.

Let \( H^k \) be a k-dimensional subspace \( k < m \).

\[
V^k_{p3}(H^k) = \left\{ v_i | v_i \in H^k \cap \Pi, v_i \text{ a p3-vertex} \right\},
\]

\[
L^k_{p3}(H^k) = \left\{ l_i | l_i \in H^k \cap \Pi, l_i \text{ a p3-line} \right\}.
\]

Definition 6. Let \( \Pi \) be a p-configuration and \( H^k \) be a k-dimensional subspace. For the set \( V \subseteq V^k_{p3}(H^k) \) the generated set \( \langle V \rangle \) is defined in the following manner:

- If \( V = \{ v_i \} \), then
  \[
  \langle V \rangle = \langle v_i \rangle := \{ v_i \}.
  \]

- If \( V = \{ v_i, v_j \} \), then
  \[
  \langle V \rangle = \langle v_i, v_j \rangle := \left\{ v_k | v_k, v_k \in l_i, l_i \in L^k_{p3}(H^k) \right\}.
  \]

- If \( |V| > 2 \), then
  \[
  \langle V \rangle := \left\{ v_k | v_k, v_k \in l_i, l_i \in L^k_{p3}(H^k), v_i \in \langle V_i \rangle, v_j \in \langle V_j \rangle, V_i, V_j \subseteq V, V_i \cap V_j = \emptyset \right\}.
  \]

The generating system of the p3-vertices \( V^k_{p3}(H^k) \) is \( G \left( V^k_{p3}(H^k) \right) \), if \( G \left( V^k_{p3}(H^k) \right) \subseteq V^k_{p3}(H^k) \) and \( \langle G \left( V^k_{p3}(H^k) \right) \rangle \subseteq V^k_{p3}(H^k) \).

By theorem 3 every vertex \( v_j \) of the p-configuration \( \Pi \) belongs to at least \( (n - 1) \) p-lines \( l_i \) and these lines generate an \( (n - 1) \)-dimensional hyperplane. Consequently a basis can be choose form these p-lines which is denoted by \( B(L^k_{n-1}) = \{ l_i | v_j \in l_i, i = 1, \ldots, n-1 \} \).

Definition 7. Consider a m-dimensional p-configuration \( \Pi \) and an \( (m-1) \)-dimensional hyperplane \( H^{m-1} \). The \( (m-1) \)-dimensional hyperplane \( H^{m-1} \) is called a supplemeter hyperplane of a p-configuration \( \Pi \), if the hyperplane \( H^{m-1} \) and every p-line of the p-configuration \( \Pi \) has at least one common point and every p-line contains at most three p-vertices (along with the new points) and there is a generating system of p3-vertices \( G \left( V^m_{p3}(H^{m-1}) \right) \) for which \( G \left( V^m_{p3}(H^{m-1}) \right) \subseteq V(L^m_{v_i}) \) for any \( v_i \notin H^{m-1} \), where \( V(L^m_{v_i}) = \{ v_k | v_k = l_i \cap H^{m-1} \text{ and } l_i \in B(L^k_{n-1}) \} \) and the interior of the convex hull of points \( V(L^m_{v_i}) \) does not contain any vertex of \( \Pi \).

Lemma 5. Let \( P_{v_i}^{m-1} \) be a facet corresponding to a vertex \( v_i \in \Pi \). If \( P_{v_i}^{m-1} \parallel Z \) for every vertex \( v_i \) of generating system \( G \left( V^m_{p3}(H^{m-1}) \right) \), then \( P_{v_k}^{m-1} \parallel Z \) for every vertex \( v_k \in V^m_{p3}(H^{m-1}) \), where \( Z \) is a hyperplane.

Proof. At first, if for every vertex \( v_p \in G \left( V^m_{p3}(H^{m-1}) \right) \) the facet \( P_{v_p}^{m-1} \) is parallel to \( Z \) for every vertex \( v_p \in \langle v_p \rangle \) then for every set \( \{ v_m, v_n \} \subseteq G \left( V^m_{p3}(H^{m-1}) \right) \) it holds that \( P_{v_q}^{m-1} \parallel Z \) for every vertex \( v_q \in \langle v_m, v_n \rangle \) because \( P_{v_q}^{m-1} \parallel Z \) and \( P_{v_q}^{m-1} \parallel Z \) and by definition 6 \( v_q, v_m, v_n \in l_i \), thus \( P_{v_q}^{m-1}, P_{v_m}^{m-1}, P_{v_n}^{m-1} \) are the facets of a 3-belt, consequently \( P_{v_q}^{m-1} \parallel Z \).

Generally suppose that for every set \( V_j \subset V_i \subseteq G \left( V^m_{p3}(H^{m-1}) \right) \) the facet \( P_{v_j}^{m-1} \) is parallel to \( Z \) for every vertex \( v_j \in \langle V_j \rangle \). We prove that \( P_{v_j}^{m-1} \parallel Z \) for every vertex \( v_j \in \langle V_j \rangle \). If \( v_i \in \langle V_j \rangle \) according to definition 6 there is a \( v_i \in \langle V_j \rangle \) such that \( V_i, V_j \subseteq V_i \) and \( V_i \cap V_j = \emptyset \), \( v_i, v_j, v_i \in l_i, l_i \in L^k_{p3}(H^k) \), thus \( P_{v_i}^{m-1}, P_{v_j}^{m-1}, P_{v_i}^{m-1} \) are the facets of a 3-belt and because of the assumptions \( P_{v_i}^{m-1} \parallel Z \) and \( P_{v_j}^{m-1} \parallel Z \) the facet \( P_{v_i}^{m-1} \) is parallel to \( Z \).

Consequently \( P_{v_k}^{m-1} \parallel Z \) for every vertex \( v_k \in V^m_{p3}(H^{m-1}) = \langle G \left( V^m_{p3}(H^{m-1}) \right) \rangle \).
Theorem 7. If an \((m - 1)\)-dimensional hyperplane \(H^{m-1}\) is a superset hyperplane of a p-configuration \(\Pi\), then there is a direction \(z\) that \(\mathcal{P} \oplus S(z)\) is a parallelotope and \(\Pi^{m-1} \subseteq H^{m-1}\) for the p-subconfiguration \(\Pi^{m-1}\) of the p-configuration \(\Pi \oplus H(z)\).

Proof. Since \(|V(L^m_i)| = |B(L^m_i)\) = \(n - 1\) and the hyperplane \(H^{m-1}\) is a superset hyperplane, \(G\left(V_{p3}^{m-1}(H^{m-1})\right) \subseteq V(L^m_i)\), thus \(G\left(V_{p3}^{m-1}(H^{m-1})\right) \leq n - 1\) and the number of the facets \(PF_{G-1}\) of \(\mathcal{P}\) corresponding to the generating system of the p3-vertices \(G\left(V_{p3}^{m-1}(H^{m-1})\right)\) is at most \((n - 1)\), therefore the intersection of these facets is an at least 1-dimensional space \(Z\). By lemma 5 every facet of \(\mathcal{P}\) corresponding to the p3-vertices of \(V_{p3}^{m-1}(H^{m-1})\) are parallel to the space \(Z\), consequently according to theorem 2 the \(\mathcal{P} \oplus S(z)\) is a parallelotope for the segment \(S(z) \in Z\).

If \(v_k \in V(L^m_i)\) and \(v_k \in \Pi\), then the facet \(PF_{m-1}\) corresponds to the p-vertex \(v_k = II^{0}0\). If \(v_i \in V(L^m_i)\) and \(v_i \notin \Pi\) then there is a p-segment \(II^{1}\), for which \(II^{1} \cap H^{m-1} = v_i\) and the \((n - 2)\)-dimensional face \(PF_{m-2}\) corresponds to the p-segment \(II^{1}\). Consider an \((n - 1)\)-dimensional support hyperplane \(SF_{m-1}\) of the \((n - 2)\)-dimensional face \(PF_{m-2}\). So \(SF_{m-1}\) corresponds to the p-segment \(II^{1}\). \(B(L^m_i)\) is a basis of the p-lines \(l_i \notin H^{m-1}, i = 1, \ldots, n - 1\), thus the intersection of all facets \(PF_{j}\) corresponding to \(II^{0} \cup l_j \subseteq L_j\) and all \((n - 1)\)-dimensional support hyperplanes \(SF_{j}\) corresponding to \(II^{1}\) is a line \(z\), where \(L_i \cup L_j = B(L^m_i)\). \(G\left(V_{p3}^{m-1}(H^{m-1})\right) \subseteq V(L^m_i)\) thus for the segment \(S(z)\) the \(\mathcal{P} \oplus S(z)\) is a parallelotope.

The facets \(PF_{j}\) and \(SF_{j}\) are equivalent to the facets \(\mathcal{P} \oplus S(z)\). Furthermore the intersection of the facets \(\mathcal{P} \oplus S(z)\) is a segment \(S(z)\). By lemma 3 \(\Pi \oplus H(z)\) containing the segment \(S(z)\) contains all vertices \(v_k \in V(L^m_i)\), consequently \(\Pi \oplus H(z)\) is \(H^{m-1}\) because both \((m - 1)\)-dimensional hyperplanes contain the same \(m\)-dimensional simplex. By lemma 4 \(\Pi \oplus H(z)\) is \(H^{m-1}\) and \(\Pi \oplus H(z)\) is \(H^{m-1}\), consequently \(\Pi^{m-1} \subseteq H^{m-1}\).

Definition 8. Consider the superset hyperplane \(H^{m-1}\) of a p-configuration \(\Pi\). The vertices of \(\Pi \cap H^{m-1}\) are defined by points \(v_i\) for which

\((V1)\) \(v_i \in \Pi\) and \(v_i \in H^{m-1}\),

\((V2)\) \(v_i = l_i \cap H^{m-1}\) where \(l_i \not\subset H^{m-1}\).

The lines of \(\Pi \cap H^{m-1}\) are defined by

\((L1)\) p-lines \(l_i \in \Pi\) and \(l_i \in H^{m-1}\),

\((L2)\) \(S \cap H^{m-1}\) where \(S\) is a 2-dimensional plane generated by two p-lines \(l_i\) and \(l_j\) of the p-configuration \(\Pi\) and \(l_i,l_j \not\subset H^{m-1}\).

Theorem 8. For the superset hyperplane \(H^{m-1}\) of a p-configuration \(\Pi\) the intersection \(\Pi \cap H^{m-1}\) is equal to the p-subconfiguration \(\Pi^{m-1}\) of the p-configuration \(\Pi \oplus H(z)\).

Proof. By theorem 7, if an \((m - 1)\)-dimensional hyperplane \(H^{m-1}\) is a superset hyperplane of \(\Pi\), then there is a direction \(z\) such that \(\mathcal{P} \oplus S(z)\) is a parallelotope and \(\Pi^{m-1} \subseteq H^{m-1}\) for the p-subconfiguration \(\Pi^{m-1}\) of \(\Pi \oplus H(z)\). According to theorem 6 and definition 8 \((P2)\) is equal to \((V1)\), \((P3)\) is equal to \((V2)\) and \((P1)\) is equal to \((L1)\), consequently it is sufficient to prove the equality for case \((L2)\).

Let \(l\) be equal to \(S \cap H^{m-1}\). By the case \((L2)\) of the definition 8 p-lines \(l_i\) and \(l_j\) generate a 2-dimensional plane \(S\). On the one hand we can choose \(l_i,l_j\) in such a way that the p-segments \(\Pi F_i^{1} \subseteq l_i\) and \(\Pi F_j^{1} \subseteq l_j\) have a common polygon \(\Pi F_2^{2} \subseteq S\). On the other hand \(l_i,l_j \subseteq S\) and \(l_i,l_j \not\subset H^{m-1}\), consequently \(l_i,l_j \not\subset \Pi^{m-1}\) by theorem 7. Thus \(l_i \cap \Pi^{m-1} = v_i\) and \(l_j \cap \Pi^{m-1} = v_j\) are p-vertices of the subconfiguration \(\Pi^{m-1}\) according to case \((P3)\) of theorem 6.

The polygon \(\Pi F_2^{2}\) corresponds to an \((n - 3)\)-dimensional face \(PF_2^{n-3}\), the p-segments \(\Pi F_1^{1}\) and \(\Pi F_j^{1}\) correspond to the \((n - 2)\)-dimensional faces \(PF_1^{n-2}, PF_j^{n-2}\), where \(\Pi F_i^{1} \subseteq \Pi F_2^{2}, \Pi F_j^{1} \subseteq \Pi F_2^{2}\).
therefore $PF_{n-3} \subseteq PF_{n-2}$ and $PF_{n-3} \subseteq PF_{n-2}$. Thus after the extraction $PF_{n-3} \oplus S(z) \subseteq PF_{n-2} \oplus S(z)$, where p-vertices $v_i$ and $v_j$ correspond to the facets $PF_{n-2} \oplus S(z)$ and $PF_{n-2} \oplus S(z)$ and a p-line $l$ corresponds to the $(n-2)$-dimensional faces $\langle PF_{n-3} \oplus S(z) \rangle$ of $\mathcal{P} \oplus S(z)$. By lemma 2 $v_i \cap v_j \in H$, consequently $l = I$. That is $l \notin \Pi$ and $l \in \Pi \oplus H(z)$, thus $l \in \Pi^{m-1}$.

Consider a p-line $l$ for which $l \notin \Pi$ and $l \in \Pi^{m-1}$. Let $H^{m-1}$ be an $(m-1)$-dimensional hyperplane of the p-subconfiguration $\Pi^{m-1}$. Thus $\Pi^{m-1} \subseteq H^{m-1}$, consequently $l \subseteq H^{m-1}$. If the segment $(\Pi \oplus H(z))F^1 \subseteq l$ and

$$\langle II \oplus H(z) \rangle F^0 \subseteq (\Pi \oplus H(z))F^1 \quad \text{and} \quad \langle II \oplus H(z) \rangle F^0 \subseteq (\Pi \oplus H(z))F^1,$$

then for the corresponding parallelootope $\mathcal{P} \oplus S(z)$ by lemma 2

$$\langle \mathcal{P} \oplus S(z) \rangle F^{n-1} \supseteq (\mathcal{P} \oplus S(z))F^{n-2} \quad \text{and} \quad \langle \mathcal{P} \oplus S(z) \rangle F^{n-2} \supseteq (\mathcal{P} \oplus S(z))F^{n-2}.$$  \hspace{1cm} (11)

Before the extraction of the parallelootope $\mathcal{P}$ the $(n-2)$-dimensional face $\langle \mathcal{P} \oplus S(z) \rangle F^{n-2}$ was $(n-3)$-dimensional face $PF_{n-3}$. For the facets $\langle \mathcal{P} \oplus S(z) \rangle F^{n-2}$, $(\mathcal{P} \oplus S(z))F^{n-2}$ there are two cases.

The facets $\langle \mathcal{P} \oplus S(z) \rangle F^{n-2}$, $(\mathcal{P} \oplus S(z))F^{n-2}$ are either $(n-2)$-dimensional faces $PF_{n-2}^1$, $PF_{n-2}^2$ or facets $PF_{n-1}^1$, $PF_{n-1}^2$. In the second case we choose $(n-2)$-dimensional faces $PF_{n-2}^1 \subseteq PF_{n-1}^1$ and $PF_{n-2}^2 \subseteq PF_{n-1}^2$ for which

$$PF_{n-2}^1 \supseteq PF_{n-2}^1 \quad \text{and} \quad PF_{n-2}^2 \supseteq PF_{n-2}^2.$$  \hspace{1cm} (12)

In the first case this relation holds. Thus for the p-configuration II by lemma 2

$$\Pi F^1 \subseteq \Pi F^2 \quad \text{and} \quad \Pi F^1 \subseteq \Pi F^2.$$  \hspace{1cm} (13)

$S \cap H^{m-1} = l$ is satisfied for $l_1 = \langle \Pi F^1 \rangle$, $l_2 = \langle \Pi F^2 \rangle$ and $S = \langle \Pi F^2 \rangle$. Q.e.d.

Consequently the p-configuration $II \oplus H(z)$ is equal to $II \cup (\Pi \cap H^{m-1})$.

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