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On extraction and projection of Dirichlet-Voronoi cells of root-lattices

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Abstract: In discrete geometry the classification of the parallelotopes is an important problem. P. ENGEL obtained 5-dimensional parallelotopes from primitive parallelotopes by contracting and extracting of the obtained minimal elements, respectively. V. Grishukhin called a parallelotope free, if it can be contracted or extracted in any direction. He proved that the Dirichlet-Voronoi cells of the dual root lattice E_6^* and infinite series of the lattices D_{2m}^+, $m \geq 4$ are nonfree in all directions. In this paper we prove that the Dirichlet-Voronoi cells of the root-lattices are free, except E_8.

Keywords: Dirichlet-Voronoi cell, Parallelotopes, Root-lattices

1 Introduction

The parallelotope $\mathcal{P}$ is a convex polytope that fills the space face to face by its translation copies without intersecting by inner points. The centers of the parallelotopes form an n-dimensional lattice. An n-dimensional parallelotope is called primitive if exactly $n+1$ adjacent parallelotopes meet in each vertex. B.A. VENKOV and later P. MCMULLEN proved the following important theorem for parallelotopes. A.D. ALEKSANDROV in [1] simplified Venkov's proof.
Theorem 1. (B.A. VENKOV [40], P. MCMULLEN [31]) *The polytope $\mathcal{P}$ is a parallelotope if and only if*
(i) $\mathcal{P}$ is centrally symmetric
(ii) each facet of $\mathcal{P}$ is centrally symmetric
(iii) the 2-dimensional orthogonal projection along any $(n-2)$-face of $\mathcal{P}$ is either a parallelogram or a centrally symmetric hexagon.

The edges of the parallelogram and the centrally symmetric hexagon of the above property (iii) are the projections of $(n-1)$ facets of $\mathcal{P}$. These facets form a 2- and 3-belt, respectively.

At first we discuss briefly the results of the classification of parallelotopes. Two parallelotopes in the plane were already well-known in the antiquity: the centrally symmetric hexagon (primitive) and the parallelogram (not primitive). E.S. FEDOROV in [16] gave the 5 combinatorically different parallelotopes in 3-dimension, among which the truncated octahedron is primitive while the others are not primitive, namely the elongated octahedron, rhombic dodecahedron, the hexagonal prism and the cube. B.N. DELONE [8] found 51 different types of 4-dimensional parallelotopes. M.I. SHTOGRIN gave the missing 52nd in [34]. 17 of these are zonotopes and the 35 others are the regular 24-cell and the Minkowski sum of this with some zonotope. An n-dimensional zonotope in $\mathbb{E}^n$ is the vector sum of m line segments, in other words, it is the image of a regular m-cube under some orthogonal projection. Further results can be seen in [30]. Out of these types three are primitive. S.S.

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RYSHKOV and E.P. BARANOWSKII [32] found 221 primitive 5-dimensional parallelotopes, which was completed by P. ENGEL and V. GRISHUKHIN [14] with another one. P. ENGEL in [12] and [13] gave 179372 combinatorically different types of 5-dimensional parallelotopes.

It is easily seen that the set $\{\text{parallelotope}, <\}$ is partially ordered with maximal and minimal elements. Actually, consider the parallelotopes $\mathcal{P}$ and $\mathcal{Q}$ and the relation $<$. There is $\mathcal{P} < \mathcal{Q}$ if and only if there exists a direction $\mathbf{z}$ for which $\mathcal{P} \oplus \lambda\mathbf{z} = \mathcal{Q}$, where Minkowski sum is denoted by $\oplus$. In this case $\mathcal{P}$ is called the contraction of $\mathcal{Q}$ and $\mathcal{Q}$ is the extraction of $\mathcal{P}$. Denote by $S(\mathbf{z})$ the segment of the direction $\mathbf{z}$ and of the length z. V. GRISHUKHIN [19] called the vector $\mathbf{z}$ free if the Minkowski sum $\mathcal{P} \oplus S(\mathbf{z})$ is again a parallelotope. A parallelotope is called free, if it has free vectors. An element is maximal if it cannot be extracted in a non-trivial way, i.e. there is no parallelotope of which it can be contracted. An element is minimal if it cannot be contracted. In 3-dimension the maximal element is the primitive truncated octahedron from which you can get the all parallelotopes using contractions. The minimal element is the cube (see [6] or [24]). In 4-dimension there are 4 maximal elements but only 3 of them are primitive. Using contractions, 2 minimal elements are found. From these two minimal elements you can get all parallelotopes by extractions ([4], [35], [18]). It is clear that from the 3 primitive elements you can not get the all parallelotopes only with contractions.

P. ENGEL obtained 5-dimensional parallelotopes from primitive parallelotopes by contracting and extracting of the obtained minimal elements, respectively. Unfortunately, in general not every parallelotope can be obtained such way because by the theorem 15 the DV cell of lattice E_8 cannot extract in any direction further it is a parallelotope of zero width in each direction thus it cannot contract. So there exists a non-primitive maximal element, which is also minimal and cannot be obtanined from primitive element in any way, i.e. the DV cell of the lattice E_8 is nonfree. V. Grishukhin [19] proved that the Dirichlet-Voronoi cells of the dual root lattice E_6^* and infinite series of the lattices $D_{2m}^+, m \geq 4$ are nonfree in all directions.

2 Dirichlet-Voronoi cells

The concept of the Dirichlet-Voronoi cell was introduced by DIRICHLET [9] and VORONOI [42]. Voronoi polytope and Voronoi cell are also used instead of Dirichlet-Voronoi cell in higher dimensions. We use shortly DV cell in this paper.

Definition 1. *Let us give a discrete point set L in the n-dimensional Euclidean space $\mathbb{E}^n$. The DV cell of a point P_i of the set L is the set of points which are at least as close to point P_i as to any other point P_j of the set L, i.e.*

$$\text{DV}(P_i) = \{x \in \mathbb{E}^n : \text{dist}(x, P_i) \leq \text{dist}(x, P_j) \textit{ for every } j\}.$$

If the point set L is a lattice it is clear that any two DV cells are translated copies of each other and any cell and its $(n-1)$-dimensional faces (called facets) are centrally symmetric convex bodies. DV cells of a lattice form a lattice tiling, that is, their union covers the space and their interiors are mutually disjoint. This tiling is face to face. In this paper we investigate only DV cells of lattices so denote the n-dimensional lattice and the DV cell of the point $P \in \Lambda^n$ by Λ^n and by $\text{DV}^n(P)$, respectively. Classical problems of DV cells can be seen for example in [5], [20], [21], [22].

By the above the DV cell is a special kind of a parallelotope. The converse statement is the Voronoi's conjecture. G.F. VORONOI asked whether each parallelotope is the affine image of a DV cell. In [42] and [43] he proved the conjecture for primitive parallelotopes. O.K. ZSITOMIRSZKIJ [44] extended G.F. VORONOI's proof to $(n-2)$-primitive parallelotopes, i.e. when each belt of the parallelotope is 3-belt. P. MCMULLEN [30] proved the conjecture for zonotopes. R.M. ERDAHL gave another proof for this in [15].

3 Root-lattices

We define the special root-lattices. The n-dimensional cube-lattice:

$$Z_n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of integer numbers. You can get the lattice A_n from the $(n+1)$-dimensional cube-lattice:

$$A_n = \{(x_0, x_1, \ldots, x_n) \in Z_{n+1} : x_0 + x_1 + \cdots + x_n = 0\}.$$

The lattice D_n is called by the checkerboard lattice:

$$D_n = \{(x_1, x_2, \ldots, x_n) \in Z_n : x_1 + x_2 + \cdots + x_n \text{ even}\}.$$

Later on it will be important the lattice E_8 and lattices E_7, E_6 given by means it:

$$E_8 = \left\{(x_1, x_2, \ldots, x_8) : x_i \in \mathbb{Z} \text{ or } x_i \in \mathbb{Z} + \frac{1}{2} \text{ for every } x_i,\ \sum x_i \equiv 0 \pmod 2\right\},$$

$$E_7 = \{(x_1, x_2, \ldots, x_8) \in E_8 : x_1 + x_2 + \cdots + x_8 = 0\},$$

$$E_6 = \{(x_1, x_2, \ldots, x_8) \in E_8 : x_1 + x_8 = x_2 + \cdots + x_7 = 0\}.$$

4 Parallelotopes

B.A. VENKOV introduced the concept of a parallelotope of non-zero width in the direction of a k-subspace X^k. A parallelotope $\mathcal{P}$ has non-zero width along X^k if the intersection $\mathcal{P} \cap (X^k + \mathbf{a})$ is either k-dimensional or empty for every translation vector $\mathbf{a}$.

Theorem 2. ([41]) *Let an n-dimensional parallelotope $\mathcal{P}$ be of non-zero width along X^k. Then*
(i) the projection of $\mathcal{P}$ along X^k is a parallelotope (of dimension $n-k$)
(ii) the lattice vectors $\mathbf{t}_i$ related to the facets F_i which are parallel to X^k generate an $(n-k)$-dimensional lattice spanned by a space X^{n-k} which is transversal to X^k.

For $k = 1$ the k-subspace X^k is a line. The direction of this line is given by a vector $\mathbf{z}$, so the z width of a parallelotope $\mathcal{P}$ along $\mathbf{z}$ is the minimal length of the intersections of lines parallel to $\mathbf{z}$ with $\mathcal{P}$. If this minimal length is equal to zero then a parallelotope $\mathcal{P}$ is of zero width in the direction $\mathbf{z}$. V. GRISHUKHIN proved that $\mathcal{P}$ has non-zero width in a direction $\mathbf{z}$ if and only if $\mathcal{P}$ has a closed edge zone parallel to $\mathbf{z}$ and $\mathcal{P}$ is the Minkowski sum of a segment $S(\mathbf{z})$ and a parallelotope $\mathcal{P}'$ of zero width in the direction $\mathbf{z}$, i.e. $\mathcal{P} = \mathcal{P}' \oplus S(\mathbf{z})$. Thus the following theorem holds:

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Theorem 3. (V. GRISHUKHIN [18]) *For any parallelotope exactly one of the following statements holds:*
(i) It is a zonotope or
(ii) it is a parallelotope of zero width in any direction or
(iii) it is the Minkowski sum of a zonotope with a parallelotope of zero width in any direction.

The Minkowski sum $\mathcal{P} \oplus S(\mathbf{z})$ is not necessarily a parallelotope. The following theorem gives necessary and sufficient conditions for this sum to be a parallelotope.

Theorem 4. (V. GRISHUKHIN [17], M. DUTOUR [10]) *The following assertions are equivalent for a parallelotope $\mathcal{P}$:*
(i) the Minkowski sum $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope
(ii) the vector $\mathbf{z}$ is orthogonal to the normal vector of at least one facet of each 3-belt of $\mathcal{P}$.

5 Projection of parallelotopes and DV cells

Definition 2. *The* shadow boundary *of a parallelotope $\mathcal{P}$ in a direction $\mathbf{z}$ consists of all boundary points $\mathbf{x}$ of $\mathcal{P}$ for which the line $\{\mathbf{x} + \lambda\mathbf{z} | \lambda \in \mathbb{R}\}$ is a support line of $\mathcal{P}$. (There is no point of the line $\{\mathbf{x} + \lambda\mathbf{z} | \lambda \in \mathbb{R}\}$ belonging to the interior of $\mathcal{P}$).*

It is well known that the shadow boundary of a convex polytope is the union of its some closed $(n-1)$- and $(n-2)$-dimensional faces. If the parallelotope $\mathcal{P} \oplus S(\mathbf{z})$ is of non zero width in a direction $\mathbf{z}$ then the shadow boundary contains only $(n-1)$-facets which are called facets parallel to $\mathbf{z}$ by B.A. VENKOV. As a generalization of this consider the following definitions. The introduction of these definitions is detailed in [23]).

Denote by F_i a facet of the parallelotope $\mathcal{P}$ and by $\mathbf{t}_i$ the lattice vector connecting the center of the parallelotope $\mathcal{P}$ with a parallelotope $\mathcal{P}_i$ where $\mathcal{P}$ and $\mathcal{P}_i$ are adjacent by the facet F_i. We name this lattice vector as the *relevant vector* of F_i.

Definition 3. *Let $\mathcal{P}$ be a parallelotope and F one of its faces of the dimension $(n-2)$. Then there exist two facets of $\mathcal{P}$, say F_1 and F_2 with relevants $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively, which contain this face. If F determines a 2-belt then we call the lattice vector $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ the its* generalized relevant vector. *(This is a lattice vector of the lattice of translations associated to F.)*

We note that there is no generalized relevent vector of the $(n-2)$-face in a 3-belt.

Definition 4. *Let $\mathcal{P}$ be a parallelotope with lattice Λ and $\mathbf{z}$ be a given direction of the space. We denote by $\Lambda_{\mathbf{z}}$ the sublattice of Λ spanned by those relevant or generalized relevant vectors of $\mathcal{P}$ whose corresponding faces are maximal ones (with respect to the face-lattice of the parallelotope) of faces belonging to the shadow boundary of a direction $\mathbf{z}$. We shall say that this lattice is the* Venkov lattice *associated to the direction $\mathbf{z}$.*

Further on, we use the following statement on the connection between the projection and the extraction of a parallelotope.

Theorem 5. (Á.G. HORVÁTH [23]) *Let $\mathcal{P}$ be a parallelotope, $\mathbf{z}$ a direction and H a hyperplane transversal to $\mathbf{z}$. Then the following statements are equivalent:*
(i) The polytope $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope
(ii) The projection of $\mathcal{P}$ along the line of $\mathbf{z}$ onto the hyperplane H is a parallelotope with respect to the projection of the Venkov lattice $\Lambda_{\mathbf{z}}$.

We remark that it follows from propety (ii) that the Venkov-lattice $\Lambda_{\mathbf{z}}$ of the direction $\mathbf{z}$ is an $(n-1)$-dimensional lattice. Next the layers of lattice Λ^n are defined:

Definition 5. *The intersection of the subspace generated by $(n-1)$ linearly independent lattice vectors and lattice Λ^n is called a layer of the lattice Λ^n and is denoted by $\Lambda^{n-1}_{(0)}$. A vector which supplements a basis of the layer $\Lambda^{n-1}_{(0)}$ to a basis of the lattice Λ^n is denoted by $\mathbf{a}_n$. The translation of the layer $\Lambda^{n-1}_{(0)}$ with the j-fold of the vector $\mathbf{a}_n$ is denoted by $\Lambda^{n-1}_{(j)}$.*

A few layers of lattice Λ^n have the following special property.

Definition 6. *If sublattice Λ^{n-1} of Λ^n has the following property: $\mathbb{R}^n \setminus \bigcup\{\mathcal{P}+\lambda_i \quad \lambda_i \in \Lambda^{n-1}\}$ has two path connected components, then the set $\bigcup\{\mathcal{P}+\lambda_i \quad \lambda_i \in \Lambda^{n-1}\} := [\Lambda^{n-1}(\mathcal{P})]$ is called a parallelotope* lamina. *The lattice Λ^{n-1} is called a* laminar lattice.

Note, however, that as it is shown later on, there are lattices which have no layer with this special property.

Definition 7. *A face of $\mathcal{P}$ which is common with another parallelotope $\mathcal{P}_i$ of the laminar lattice Λ^{n-1} is called a* connecting face *of $\mathcal{P}$ if it is a maximal common face with respect to the face-lattice of $\mathcal{P}$.*

It is easy to see that the dimension of such a face is either $(n-1)$ or $(n-2)$. If we investigate DV cells instead of parallelotopes, then the relevant vectors are facet vectors, too. So for the DV cells the relevant vectors and generalized relevant vectors are called generalized facet vectors in brief (see: [38]).

Let $P \in \Lambda^n$ be a point and $\mathbf{z}$ be a direction. Intersect lattice Λ^n with a $(n-1)$-dimensional hyperplane H which contains point P and is perpendicular to $\mathbf{z}$. Denote the resulting set of intersection points $\Lambda^n \cap H$ by Λ^{n-1}, if it is an $(n-1)$-dimensional lattice.

Theorem 6. (A. VÉGH [38]) *The following statements are equivalent for the DV cell $\mathrm{DV}^n(P)$ and the vector $\mathbf{z}$:*
(i) the orthogonal projection of the cell $\mathrm{DV}^n(P)$ to the $(n-1)$-dimensional hyperplane H along $\mathbf{z}$ is an $(n-1)$-dimensional DV cell $\mathrm{DV}^{n-1}(P)$ of the lattice Λ^{n-1}, where $\Lambda^{n-1} = \Lambda^n \cap H$.
(ii) the vector $\mathbf{z}$ is orthogonal to at least one generalized facet vector of each 2- and 3-belt.
(iii) $\mathbb{R}^n \setminus [\Lambda^{n-1}(\mathrm{DV}^n(P))]$ has two path connected components. (By the definition of the parallelotope lamina $[\Lambda^{n-1}(\mathrm{DV}^n(P))] = \bigcup\{\mathrm{DV}^n(P)+\lambda_i \quad \lambda_i \in \Lambda^{n-1}\}$)

We remark that if the property (ii) does not hold then the orthogonal projection of a DV cell to H can still be a DV cell but it is clear that in this case properties (i) and (iii) do not hold, either. One example is the orthogonal projection of the regular hexagon prism in the direction $\mathbf{z}$ where the vector $\mathbf{z}$ is a normal vector of the lateral face.

6 Extraction of parallelotopes and DV cells

In this section we investigate the connection between the extraction of parallelotopes and the coordinates of relevant vectors.

Lemma 1. (A. VÉGH [39]) *If a parallelotope $\mathcal{P}$ can be extracted in a direction* $\mathbf{z}$*, then there exists a basis* $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ *of the lattice in which* $[\Lambda_z] = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{n-1}]$ *holds and the* n*th coordinates of the relevant vectors which do not belong to the Venkov-lattice* Λ_z *are* ± 1.

Namely, if a parallelotope $\mathcal{P}$ can be extracted in a direction $\mathbf{z}$ then the Venkov-lattice is a primitive $(n-1)$-dimensional sublattice of Λ and all other relevant vectors have ± 1 nth coordinates in case of a suitable vector $\mathbf{e}_n$ completing to a basis.

Lemma 2. (A. VÉGH [39]) *If there exists a basis in which the coordinates of relevant vectors of a parallelotope* $\mathcal{P}$ *are* $0, \pm 1$ *and the parallelotope* $\mathcal{P}$ *is an affine image of a DV cell, then there exists a direction* $\mathbf{z}$ *such that* $\mathcal{P} \oplus S(\mathbf{z})$ *is a parallelotope, too.*

Remark that by Theorem 5 and in case the above conditions hold the projection of a parallelotope $\mathcal{P}$ in the direction $\mathbf{z}$ is a parallelotope of the Venkov-lattice $\Lambda_{\mathbf{z}}$. On investigating DV cells instead of parallelotopes it can be seen that the projection is a DV cell, i.e. the following lemma holds.

Lemma 3. (A. VÉGH [39]) *If there exists a basis in which the relevant vectors of the DV cell* $\mathcal{D}$ *have coordinates* $0, \pm 1$*, then there exists a direction* $\mathbf{z}$ *such that the projection of the cell* $\mathcal{D}$ *in the direction* $\mathbf{z}$ *is the DV cell of the lattice* $\Lambda_{\mathbf{z}}$.

7 Minimal vectors and perfect forms

Let $\mathrm{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \{\mathbf{a}_i\}$ be a basis of Λ. The minimum $m(\Lambda)$ of the lattice Λ is defined by

$$m(\Lambda) \in \mathbb{R}^+ : m(\Lambda) = |\mathbf{m}| \leq |\mathbf{v}| \text{ for an } \mathbf{m} \in \Lambda \setminus \{\mathbf{0}\} =: \dot{\Lambda} \text{ and for any } \mathbf{v} \in \dot{\Lambda}.$$

We may assume (by a similarity of $\mathbb{E}^n$) that $m(\Lambda) = 1$. The set of the minimum vectors is called the minima of Λ and is denoted by

$$\mathbf{M}(\Lambda) := \{\mathbf{m} \in \Lambda : |\mathbf{m}| =: m(\Lambda) =: 1\}.$$

The maximal A-coordinate of the minima of Λ is defined by

$$L(\mathrm{A}) := \max\left\{x_i \in \mathbb{Z} : \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{m}, \mathbf{m} \in \mathbf{M}(\Lambda)\right\} \in \mathbb{N}.$$

Consider the minimum of these maximal A-coordinates of the minima of Λ by changing basis A in Λ, i.e. define

$$L(\Lambda) := \min\{L(\mathrm{A}) \in \mathbb{N} : \mathrm{A} \text{ is any basis of } \Lambda\}.$$

Finally, vary the lattices Λ in $\mathbb{E}^n$. Then

$$L_n := L(\mathbb{E}^n) := \max\left\{L(\Lambda) \in \mathbb{N} : \Lambda \text{ is any lattice of } \Lambda \in \mathbb{E}^n\right\}.$$

In general, the problem is to give L_n. In other words in any lattice of $\mathbb{E}^n$ find a basis, in which the maximal coordinate of the minima of the lattice is the possible smallest.

Consider a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ of the lattice Λ in an orthonormalized basis of $\mathbb{R}^n$. Denote by A the matrix of the coordinates of the basis vectors. Using the matrix A you can define the positive definite quadratic form of the lattice Λ:

$$Q(\mathbf{x}) = \langle A\mathbf{x}, A\mathbf{x}\rangle = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T G\mathbf{x}, \mathbf{x} \in \mathbb{Z}^n,$$

where $G = A^T A$ is a Gram matrix. The positive definite quadratic form Q depends on basis A of the lattice Λ. Let A and B be any two bases of the lattice Λ, then there is a unimodular matrix U for which $A = UB$. On the other hand, different lattices belong to a given quadratic form Q . For example, suppose that a lattice Λ is generated by a basis A and a lattice Λ' is generated by a basis A'. But in this case there is an orthogonal transformation O for which $A = OA'$. Thus it is shown that quadratic forms provide an alternative language for studying lattices. Here we discuss only the most important results about quadratic forms. Other definitions and results can be found in [20], [5], [29]. Similarly to the minimum of the lattice Λ the so-called homogeneous minimum of the quadratic form Q can be defined as follows:

$$m(Q) := min\{Q(\mathbf{z}) : \text{where} \quad \mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\}.$$

Let $\mathrm{M}(Q)$ be the set of points $\mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, for which the value $Q(\mathbf{z})$ is minimal.

Definition 8. *A positive definite quadratic form Q is called* perfect, *if it is determined uniquely by the equations $Q(\mathbf{z}_i) = m(Q)$, where $\mathbf{z}_i \in \mathrm{M}(Q)$ are minimal vectors in $\mathbb{Z}^n$, i.e. if there is no quadratic form $Q_1 \neq Q$ with $Q_1(\mathbf{z}_i) = m(Q)$, where $\mathbf{z}_i \in \mathrm{M}(Q)$.*

L_n is determined for $n \leq 5$ in [21], where the unique existence and the increasing of L_n by n are also discussed.

Theorem 7. (Á.G. HORVÁTH [21]) $L_n = 1$ *for* $n \leq 5$.

By investigating admissible extensions of lattices and classifying the lattices according to their indices we prove in [36] that L_6 is equal to one, i.e. for every Euclidean 6-lattice Λ there is a basis in which the maximal coordinate of all the minimum vectors of Λ is equal to at most 1.

Theorem 8. (A. VÉGH [36]) $L_6 = 1$.

Á.G. HORVÁTH conjectures that $L_8 = 2$, he has proved $L_8 > 1$ by the famous lattice E_8 in $\mathbb{E}^8$ and he has proved the following for root-lattices:

Theorem 9. (Á.G. HORVÁTH [22]) $L(Z_n) = L(A_n) = L(D_n) = L(E_6) = L(E_7) = 1$ *and* $L(E_8) = 2$.

Perfect forms are known in the $n \leq 7$ dimensions. In the euclidean plane J.L. LAGRANGE [28] dealt with this question and he found one perfect form. In the space there is also one perfect form, given by C.F. GAUSS [11]. A. KORKINE and G. ZOLOTAREFF [26], [27] revealed in the 4 and 5 dimensions 2 and 3 perfect forms, respectively. E.S. BARNES [2] found 7 different perfect forms in the 6 dimension. K.C. STACEY [33] investigated 7 dimensional perfect forms, but he omitted one of the 33, which J.H. CONWAY- N.J.A SLOANE [7] completed, finally D.O. JAQUET-CHIFFELLE [25] proved the completeness of these. J. MARTINET found more than 10916 perfect forms in 8 dimension. You can find the perfect forms for example in J.H. CONWAY- N.J.A SLOANE's paper [7], and in C. BATUT and J. MARTINET's works [29], [3].

Theorem 10. (D.O. JAQUET-CHIFFELLE [25]) *In [29], [3] the list of Gram matrix of the 7-dimensional perfect forms is complete.*

G.F.VORONOI's theorem of [42] about the minimum of perfect forms has a fundamental role in determining L_7.

Theorem 11. (G.F. VORONOI [42]) *For every positive definite quadratic form $Q(\mathbf{x})$ there is a perfect form $Q^*(\mathbf{x})$ such that*

$$\mathrm{M}(Q) \subseteq \mathrm{M}(Q^*).$$

Thus it is sufficient to investigate the Gramian of perfect quadratic forms in [3]. The connection between a Gram matrix and the coordinates of minimal vectors is given below:

Lemma 4. (A. VÉGH [37]) *If m is the length of the minimal vector $\mathbf{m}$, $x_1, x_2, \ldots, x_n$ are the coordinates of $\mathbf{m}$ in an arbitrary basis of the lattice, $D = det(G)$ and D_i is subdeterminant of the element a_{ii} of the Gramian (in the basis of the lattice), then $|x_i| \leq m\sqrt{\frac{D_i}{D}}$.*

Further, it is well-known from linear algebra that the matrix of the transformation of adding the c-fold of the ith row of a matrix to the jth row is unimodular. On the other hand, the conjugate of Gram matrix G of a lattice Λ with unimodular matrix U is a Gram matrix G' of a lattice Λ in a new basis. So the above transformation of the lattice Λ for Gram matrix G is equivalent to the transformation of adding the c-fold of the ith row of the Gramian G to the jth row and in the new matrix adding the c-fold of the ith column to the jth column. Using suitable transformations I proved that the coordinates of minimal vectors of the lattice are smaller than 2 in every case, thus followig theorem holds:

Theorem 12. (A. VÉGH [37]) *L_7 is equal to one, i.e. to every Euclidean 7-lattice Λ there is a basis in which the maximal coordinate of all the minimum vectors of Λ is equal to 1, at most.*

8 The DV cells of root-lattices

We investigate the DV cells of root-lattices and we use the following theorems.

Theorem 13. (J.H.CONWAY, N.J.A.SLOANE [5]) *For any root-lattice Λ the DV cell around the origin is the union of the images of the fundamental simplex under a finite reflection group or a Weyl group $W(\Lambda)$.*

The facets of the DV cell are the images of the facets not belonging to the origin of the fundamental simplex under a Weyl group $W(\Lambda)$. Thus Theorem 14 follows from Theorem 13.

Theorem 14. (J.H.CONWAY, N.J.A.SLOANE [5]) *The relevant vectors of a DV cell in a root-lattice are precisely the minimal vectors.*

So we proved the followig theorem.

Theorem 15. *For the DV cell $\mathcal{D}$ (and for its any affin image $\mathcal{P}$) of any root-lattice except the lattice E_8, there exists a direction* $\mathbf{z}$ *such that $\mathcal{D} \oplus S(z)$ ($\mathcal{P} \oplus S(z')$) is a parallelotope, i.e. the DV cells of any root-lattice are free except the DV cell of the lattice E_8, it is nonfree.*

We remark that the importance of the theorem lies in fact that for the DV cell of the lattice E_8 there is no direction for which it can be extracted.

Proof. By Theorem 9 for the root-lattices Z_n, A_n, D_n, E_6, E_7 there exists a basis in which the coordinates of minimal vectors of these lattices are $0, \pm 1$. By Theorem 14 for the root-lattices the relevant vectors of a DV cell are precisely the minimal vectors. Therefore, by Lemma 2 for the DV cell $\mathcal{D}$, resp. its any affine image $\mathcal{P}$ of any above root-lattices there exists a direction $\mathbf{z}$ such that $\mathcal{D} \oplus S(\mathbf{z})$ ($\mathcal{P} \oplus S(\mathbf{z}')$) is a parallelotope.

In the following we prove that the DV cell of the lattice E_8 cannot be extracted, namely there is no direction $\mathbf{z}$ for which DV$(E_8) \oplus S(\mathbf{z})$ is a parallelotope. In fact, by Theorem 9 there is no basis of the lattice E_8 in which the coordinates of every minimal vector are $0, \pm 1$. So, using Theorem 14, there is a relevant vector $\mathbf{x}$ in every basis of the lattice E_8 where the coordinate $x_i \neq 0, \pm 1$ for any $i = 1, ..., 8$. Suppose that the DV cell DV(E_8) can be extracted in a direction $\mathbf{z}$. Then the Venkov lattice $\Lambda_{\mathbf{z}}$ of the direction $\mathbf{z}$ is an $(n-1)$-dimensional. Choose a basis for which $[\Lambda_z] = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_7]$. So by Lemma 1 the absolute value of the 8th coordinates of every relevant vector can not be more than one. Consequently, $\mathbf{x} \in \Lambda_{\mathbf{z}}$ for the relevant vector $\mathbf{x}$ where the coordinate $x_i \neq 0, \pm 1$ for any $i = 1, ..., 8$. The relevant vectors of the DV cell DV(E_8) in $\Lambda_{\mathbf{z}}$ are the minimal vectors in the 7-dimensional Venkov lattice $\Lambda_{\mathbf{z}}$ too, because the Venkov lattice $\Lambda_{\mathbf{z}}$ is a sublattice of E_8 so the minimum cannot decrease. By Theorem 12 in every 7-dimensional lattice you can select a basis for which the coordinates of the minimal vectors, i.e. the coordinates of the relevant vectors in $\Lambda_{\mathbf{z}}$ are $0, \pm 1$. This is a contradiction. So there is no direction $\mathbf{z}$ for which DV$(E_8) \oplus S(z)$ is a parallelotope, i.e DV cell DV(E_8) is nonfree.

A similar theorem can be easily verified for the projection of the DV cells of any root-lattices:

Theorem 16. *For the DV cell $\mathcal{D}$ of any root-lattice except the lattice E_8 there exists a direction* $\mathbf{z}$ *such that the projection of the DV cell $\mathcal{D}$ in the direction* $\mathbf{z}$ *is a DV cell of the lattice $\Lambda_{\mathbf{z}}$.*

Proof. Similarly to the above, by Theorem 9 there is a basis of the root-lattices Z_n, A_n, D_n, E_6, E_7 in which the coordinates of the minimal vectors of these lattices are $0, \pm 1$. As by Theorem 14 the relevant vectors of the DV cells of the root-lattices are precisely the minimal vectors, using Lemma 3 for the DV cells of the above root-lattices there is a direction $\mathbf{z}$ for which the projection of the DV cell $\mathcal{D}$ in the direction $\mathbf{z}$ is the DV cell of the Venkov-Lattice $\Lambda_{\mathbf{z}}$.

As by Theorem 15 there is no such a direction $\mathbf{z}$ of the DV cell DV(E_8) for which it can be extracted, by Theorem 4, there is no direction $\mathbf{z}$ which is perpendicular to at least one facet vector of every 3-belt of the DV cell. So the condition (ii) of Theorem 6 does not hold. Consequently, there is no direction $\mathbf{z}$ of the DV cell DV(E_8) for which the projection of the DV cell is the DV cell of the lattice $\Lambda_{\mathbf{z}}$.

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