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ON EXTRACTION OF P-CONFIGURATIONS

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Abstract
Consider a parallelotope $\mathcal{P}$ and its dual polytope $\mathcal{P}^*$. The parallelotope configuration or the p-configuration is a system of lines and points projected vertices and edges of the polytope $\mathcal{P}^*$ from the center of the polytope $\mathcal{P}^*$ to a special $(n-1)$-dimensional hyperplane. The extraction of the parallelotope defines an extraction of the p-configuration, as well. In this paper we examine properties of the extraction of the p-configuration.

1 Configurations

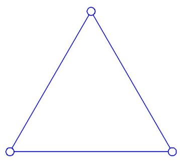
I have described the properties of the parallelotopes and of the p-configurations in details in [7]. In the current paper I will briefly summarize the most important concepts and I will investigate the extraction of the p-configurations.

A configuration is a system of p points and g straight lines arranged in a plane in such a way that every point of the system is incident with a fixed number γ of straight lines and every straight line of the system is incident with a fixed number π of points. Notation: (p_γ, g_π).

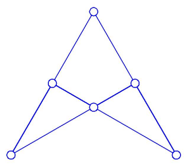
The following relation must be true for every configuration:

$$p \cdot \gamma = g \cdot \pi. \tag{1}$$

The configurations in which the number of points is equal to the number of lines, i.e. for which $p = g$ and consequently $\gamma = \pi$ are called symmetric or balanced configurations. For such a configuration the notation p_γ is used by [3], [6].



(3_2) *symmetric*



$(6_2, 4_3)$ *nonsymmetric*

Figure 1. Configurations in the plane

2 Parallelotopes

The *parallelotope* $\mathcal{P}$ is a convex polytope whose translated copies tile the space in a face to face manner. The centers of the parallelotopes form an n-dimensional lattice. In the plane there are two types of parallelotopes: parallelograms and centrally symmetric hexagons.

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Parallelotopes were characterized by B. A. Venkov[8] and later P. McMullen[5] in the following way: a polytope $\mathcal{P}$ is a parallelotope if and only if $\mathcal{P}$ is centrally symmetric, each facet of $\mathcal{P}$ is centrally symmetric, and the 2-dimensional orthogonal projection along any $(n-2)$-face of $\mathcal{P}$ is either a parallelogram or a centrally symmetric hexagon.

The edges of the parallelogram and the centrally symmetric hexagon are projections of the facets of the parallelotope $\mathcal{P}$. These facets form a 4- or a 6-*belt*, respectively.

B. A. Venkov introduced the concept of the parallelotope of non-zero width in the direction of a k-subspace X^k. A parallelotope $\mathcal{P}$ has non-zero width along X^k if the intersection $\mathcal{P}\cap(X^k+\mathbf{a})$ is either k-dimensional or empty for every translation vector $\mathbf{a}$. Denote by F^{n-1} a facet of the parallelotope $\mathcal{P}$ and by $\mathbf{t}$ the lattice vector between the centers of the two nearest parallelotopes $\mathcal{P}$ and $\mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ have the common facet F^{n-1}. This lattice vector is called *relevant vector* of the facet F^{n-1} by [2].

Theorem 1. (B. A. VENKOV [9]) *Let $\mathcal{P}$ be an n-dimensional parallelotope of non-zero width along X^k. Then the projection of $\mathcal{P}$ along X^k is a parallelotope (of dimension $n-k$) and the lattice vectors $\mathbf{t}_i$ related to the facets F_i^{n-1} which are parallel to X^k generate an $(n-k)$-dimensional lattice spanning a space X^{n-k} which is transversal to X^k.*

By this theorem for $k=n-2$ relevant vectors of a 4- or 6-belt span a 2-dimensional lattice, thus these relevant vectors are in a plane.

Consider the parallelotopes $\mathcal{P}$ and $\mathcal{Q}$ of dimension n. For $k=1$ denote by $S(\mathbf{z})$ the segment X^k of the direction $\mathbf{z}$ and of the length z. If there exists a direction $\mathbf{z}$ for which $\mathcal{P}\oplus S(\mathbf{z})=\mathcal{Q}$, where $\oplus$ denotes the Minkowski sum, then $\mathcal{P}$ is called the *contraction* of $\mathcal{Q}$ and $\mathcal{Q}$ is the *extraction* of $\mathcal{P}$. The following theorem provides an important condition for the extraction of the parallelotope $\mathcal{P}$ to be a parallelotope.

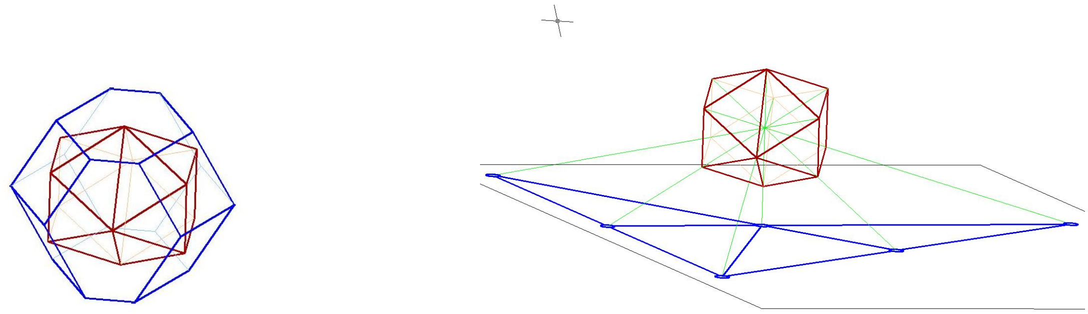
Theorem 2. [1] *Let $\mathcal{P}$ be a parallelotope and $\mathbf{z}$ be a vector. $\mathcal{P}\oplus S(\mathbf{z})$ is a parallelotope if and only if $\mathbf{z}$ is parallel to at least one facet of each 6-belt.*

Definition 1. *The* shadow boundary *of a parallelotope $\mathcal{P}$ in the direction $\mathbf{z}$ consists of all boundary points $\mathbf{x}$ of $\mathcal{P}$ for which the line $\{\mathbf{x}+\lambda\mathbf{z}|\lambda\in\mathbb{R}\}$ is a support line of $\mathcal{P}$. (There is no point of the line $\{\mathbf{x}+\lambda\mathbf{z}|\lambda\in\mathbb{R}\}$ belonging to the interior of $\mathcal{P}$). It is denoted by $sh_{\mathbf{z}}(\mathcal{P})$.*

It is well known that the shadow boundary of a convex polytope is the union of its several $(n-1)$- and $(n-2)$-dimensional closed faces [4].

3 The p-configurations

Every n-polytope has a dual structure. It can be obtained by interchanging its vertices and facets, edges and $(n-2)$-dimensional faces, and so on, generally interchanging its $(j-1)$-dimensional elements and $(n-j)$-dimensional elements, preserving coincidence between elements. We use duality for parallelotopes. The dual polytope of the parallelotope P is denoted by P^*. Facets of a parallelotope $\mathcal{P}$ are centrally symmetric, thus we can define vertices of dual polytope P^* by centers of facets of the parallelotope $\mathcal{P}$. For example consider the blue truncated octahedron in the picture 4. The dual polytope is the red polytope, which is not a parallelotope.



Duality *p-configuration*

Figure 2. Parallelotope configuration

Consider the dual polytope $\mathcal{P}^*$ of a parallelotope $\mathcal{P}$. The parallelotope configuration or the p-configuration is a system of lines and points projected vertices and edges of the polytope $\mathcal{P}^*$ from the center of the polytope $\mathcal{P}^*$ to an $(n-1)$-dimensional hyperplane, which is parallel to a hyperplane containing the center of the parallelotope $\mathcal{P}$ and not containing any vertex of the dual polytope $\mathcal{P}^*$. In this manner every parallelotope $\mathcal{P}$ determines a p-configuration denoted by Π. A line of a p-configuration Π is called p-line. A vertex of a p-configuration Π is called p-vertex.

The notation of the p-configuration is

$$((p_1)_{\gamma_1}, (p_2)_{\gamma_2}, \ldots, (p_k)_{\gamma_k}; ((g_1)_{\pi_1}, (g_2)_{\pi_2}, \ldots, (g_l)_{\pi_l}), \tag{2}$$

where p_i is the number of points which belong to γ_i lines and g_i is the number of lines which contain π_i points. The sum of p_i is equal to the number of all points of the p-configuration and the sum of g_i is equal to the number of all lines of the p-configuration.

I proved the following properties of the p-configuration in [7].

Theorem 3. *Every p-line contains two or three points of the p-configuration* Π*. Every point of the p-configuration* Π *belongs to at least* $(n-1)$ *straight lines, these lines generate a* $(n-1)$*-dimensional hyperplane.*

Theorem 4. *The intersection of two uniplanar p-line is a point of the p-configuration* Π*.*

Definition 2. *The p-configurations* Π_1 *and* Π_2 *are combinatorially equivalent, if there is a bijection between the p-vertices and the p-lines of the p-configurations* Π_1 *and* Π_2 *which preserves the point-line coincidence. We denote combinatorially equivalent p-configurations by* $\Pi_1 \cong \Pi_2$*.*

4 Parallelotopes and p-configurations

The pair of two centrally symmetric k-dimensional faces of the parallelotope $\mathcal{P}$ is denoted by PF^k and the set of parallel k-dimensional faces of the parallelotope $\mathcal{P}$ is denoted by $\langle PF^k \rangle$. Similarly the pair of two centrally symmetric k-dimensional faces of the dual polytope P^* of the parallelotope P is denoted by P^*F^k. If it is not ambiguous, the two opposite k-dimensional faces PF^k are called k-dimensional face. Similarly two opposite vertices PF^0 are called vertex of the parallelotope $\mathcal{P}$.

Consider a k-dimensional convex polytope. If all vertices and all edges of the polytope are p-vertices and p-segments of the p-configuration Π and the interior of the polytope does not contain p-vertices or p-segments, then the system of p-vertices and p-segments of the polytope is called k-dimensional face of the p-configuration Π and it is denoted by ΠF^k. We remark that the convex polytope is defined in projective sense. The k-dimensional subspace containing the face ΠF^k is denoted by $\langle \Pi F^k \rangle$ and the $(n-1)$-dimensional hyperplane containing the p-configuration Π is denoted by $\langle \Pi \rangle$.

Lemma 1. *Consider an $(n-1)$-dimensional p-configuration Π corresponding to an n-dimensional parallelotope $\mathcal{P}$. There is a bijection between an $(n-k)$-dimensional face PF^{n-k} of the parallelotope $\mathcal{P}$ and a $(k-1)$-dimensional face ΠF^{k-1} of the p-configuration Π.*

Proof. By the duality there is a bijection between the $(n-k)$-dimensional face PF^{n-k} of the parallelotope $\mathcal{P}$ and the $(k-1)$-dimensional face $\mathcal{P}^*F^{k-1}$ of the dual polytope $\mathcal{P}^*$. The central projection maps a $(k-1)$-dimensional face $\mathcal{P}^*F^{k-1}$ of the dual polytope $\mathcal{P}^*$ to a $(k-1)$-dimensional polytope F^{k-1}. The central projection preserves the convexity (in projective sense) and the coincidence. There is a one-to-one correspondence between the vertices and edges of the $(k-1)$-dimensional face $\mathcal{P}^*F^{k-1}$ of $\mathcal{P}^*$ and the vertices and edges of the $(k-1)$-dimensional polytope F^{k-1} and of Π. Thus the $(k-1)$-dimensional polytope F^{k-1} is a $(k-1)$-dimensional face ΠF^{k-1} of Π.

In other direction if ΠF^{k-1} is a $(k-1)$-dimensional face of Π then vertices of ΠF^{k-1} are the central projections of vertices of the dual polytope $\mathcal{P}^*$. The interior of the convex hull C of these vertices contains only boundary points of $\mathcal{P}^*$ otherwise the interior of ΠF^{k-1} contains p-vertices or p-segments of Π, as well. Thus the convex hull C is a $(k-1)$-dimensional face of Π because if the dimension of the convex hull C is not $(k-1)$, then the dimension of the face of Π is different, as well. A $(n-k)$-dimensional face of the parallelotope $\mathcal{P}$ corresponds to this $(k-1)$-dimensional face of Π. Consequenly there is a bijection between the $(n-k)$-dimensional face PF^{n-k} and the $(k-1)$-dimensional face ΠF^{k-1}.

Further on ΠF^{k-1} and PF^{n-k} denote the corresponding faces of the p-configuration Π and of the parallelotope $\mathcal{P}$.

Lemma 2. *Consider an $(n-k)$-dimensional face PF^{n-k} and an $(n-j)$-dimensional face PF^{n-j} of the parallelotope $\mathcal{P}$ and the corresponding $(k-1)$-dimensional face ΠF^{k-1} and $(j-1)$-dimensional face ΠF^{j-1} of the $(n-1)$-dimensional p-configuration Π. $PF^{n-j} \subseteq PF^{n-k}$ if and only if $\Pi F^{k-1} \subseteq \Pi F^{j-1}$.*

Proof. By the property of the duality $PF^{n-j} \subseteq PF^{n-k}$ is satisfied if and only if $\mathcal{P}^*F^{k-1} \subseteq \mathcal{P}^*F^{j-1}$ where $\mathcal{P}^*F^{k-1}$ and $\mathcal{P}^*F^{j-1}$ are faces of the dual polytope $\mathcal{P}^*$. The central projection preserves the relation $\subseteq$, consequently $\mathcal{P}^*F^{k-1} \subseteq \mathcal{P}^*F^{j-1}$ is equivalent to $\Pi F^{k-1} \subseteq \Pi F^{j-1}$.

Lemma 3. *Consider an $(n-j)$-dimensional face PF^{n-j} and all $(n-k)$-dimensional faces PF_i^{n-k} of the parallelotope $\mathcal{P}$ for which $PF^{n-j} \subseteq PF_i^{n-k}$ and consider the corresponding $(j-1)$-dimensional face ΠF^{j-1} and $(k-1)$-dimensional faces ΠF_i^{k-1} of the $(n-1)$-dimensional p-configuration Π. $PF^{n-j} = \bigcap\limits_i PF_i^{n-k}$ if and only if $\Pi F^{j-1} = \operatorname{conv}\left(\Pi F_i^{k-1}\right)$.*

Proof. Similarly to lemma 2 it is easy to see that the statement is satisfied by the property of the duality and the central projection. We remark that the faces ΠF_i^{k-1} are all $(k-1)$-dimensional faces of the $(j-1)$-dimensional face ΠF^{j-1} for which $\Pi F_i^{k-1} \subseteq \Pi F^{j-1}$. Q.e.d.

By the definition of the p-configuration it can easily be seen that there is a bijection between a point $\left\langle \Pi F^0 \right\rangle$ of Π and a facet $\left\langle PF^{n-1} \right\rangle$ of $\mathcal{P}$ and between a segment ΠF^1 of Π and an $(n-2)$-dimensional face PF^{n-2} of $\mathcal{P}$. Similarly according to theorem 3 a 1-dimensional line $\left\langle \Pi F^1 \right\rangle$ corresponds in a one-to-one manner to parallel $(n-2)$-dimensional faces $\left\langle PF^{n-2} \right\rangle$.

By lemma 1 there is a bijection an $(n-1)$-dimensional face ΠF^{n-1} of Π and the vertex PF^0 of $\mathcal{P}$. The set $\left\langle PF^0 \right\rangle$ consists of all vertices of the parallelotope $\mathcal{P}$, thus the set of all facets $\mathcal{P}^*F^{n-1}$ of the dual polytope $\mathcal{P}^*$ corresponds to the set $\left\langle PF^0 \right\rangle$. The central projection of all facets $\mathcal{P}^*F^{n-1}$ of the dual polytope $\mathcal{P}^*$ fills the $(n-1)$-dimensional hyperplane without gaps. Consequently there is a one-to-one correspondence between the $(n-1)$-dimensional hyperplane $\left\langle \Pi F^{n-1} \right\rangle$ of Π and vertices $\left\langle PF^0 \right\rangle$ of $\mathcal{P}$ and therefore $\left\langle \Pi F^{n-1} \right\rangle = \langle \Pi \rangle$.

These special cases are generalized in the following lemma.

Lemma 4. *Let $\mathcal{P}$ be an n-dimensional parallelotope of non-zero width along X^{n-k} and let the elements of the set $\left\langle PF^{n-k} \right\rangle$ be parallel to X^{n-k}. There is a bijection between the set $\left\langle PF^{n-k} \right\rangle$ of the parallelotope $\mathcal{P}$ and the set of a $(k-1)$-dimensional subspace $\left\langle \Pi F^{k-1} \right\rangle$ of the p-configuration Π for which $\left\langle \Pi F^{k-1} \right\rangle \cap \Pi$ is a $(k-1)$-dimensional p-configuration Π^{k-1}.*

Proof.

Generally by theorem 1 the lattice vectors $\mathbf{t}_i$ related to the facets PF_i^{n-1} which are parallel to X^{n-k} generate a k-dimensional lattice spanning a space X^k, where $i \in I$. The center points of these facets PF_i^{n-1} generate the dual polytope D^k of a k-dimensional parallelotope $\mathcal{P}^k$. By the definition of the p-configuration the central projection of the polytope D^k is a $(k-1)$-dimensional p-configuration Π^{k-1} and $\Pi^{k-1} \subseteq \Pi$.

By theorem 1 the space X^k does not contain the center points of the facets which are not parallel to X^{n-k}, thus the $(k-1)$-dimensional hyperplane $\left\langle \Pi^{k-1} \right\rangle$ does not contain other p-line or p-vertex of the p-configuration Π, consequently

$$\left\langle \Pi^{k-1} \right\rangle \cap \Pi = \Pi^{k-1}. \tag{3}$$

Consider an arbitrary $(n-k)$-dimensional face $PF^{n-k} \in \left\langle PF^{n-k} \right\rangle$. By lemma 1, a $(k-1)$-dimensional face ΠF^{k-1} of Π corresponds to an $(n-k)$-dimensional face PF^{n-k} of $\mathcal{P}$. If $PF^{n-k} \subseteq PF_j^{n-1}$ for the facets PF_j^{n-1} where $j \in J \subseteq I$, then $\Pi F_j^0 \subseteq \Pi F^{k-1}$. On the other hand every vertex $\Pi F_j^0 \in \Pi^{k-1}$, therefore $\Pi F^{k-1} \subseteq \Pi^{k-1}$, consequently $\left\langle \Pi F^{k-1} \right\rangle = \left\langle \Pi^{k-1} \right\rangle$ because both sides are $(k-1)$-dimensional hyperplanes. Thus

$$\left\langle \Pi F^{k-1} \right\rangle \cap \Pi = \left\langle \Pi^{k-1} \right\rangle \cap \Pi. \tag{4}$$

Consequently $\left\langle \Pi F^{k-1} \right\rangle \cap \Pi$ is a $(k-1)$-dimensional p-configuration Π^{k-1}.

Let $\overline{\Pi F}^{k-1}$ be an arbitrary $(k-1)$-dimensional face of Π. If $\overline{\Pi F}^{k-1} \subseteq \left\langle \Pi F^{k-1} \right\rangle$ and $\left\langle \Pi F^{k-1} \right\rangle \cap \Pi = \Pi^{k-1}$, then $\overline{\Pi F}^{k-1} \subseteq \Pi^{k-1}$ and the facets $\overline{PF}_l^{n-1}$ corresponding to all vertices $\overline{\Pi F}_l^0$ of $\overline{\Pi F}^{k-1}$ are parallel to X^{n-k}. According to lemma 3 $\overline{PF}^{n-k} = \bigcap \overline{PF}_l^{n-1}$ for all l, thus $\overline{PF}^{n-k}$ is parallel to X^{n-k}, consequently $\overline{PF}^{n-k} \in \left\langle PF^{n-k} \right\rangle$. Therefore there is a bijection between the set $\left\langle PF^{n-k} \right\rangle$ parallel to X^{n-k} and the $(k-1)$-dimensional hyperplane $\left\langle \Pi F^{k-1} \right\rangle$ for which $\left\langle \Pi F^{k-1} \right\rangle \cap \Pi = \Pi^{k-1}$. Q.e.d.

5 Extraction of p-configuration

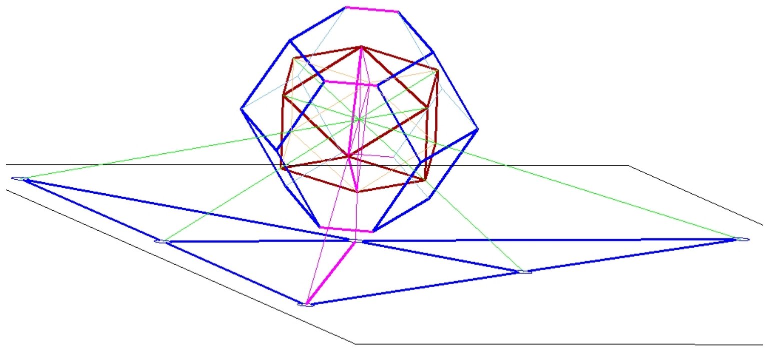
Let Π be a p-configuration corresponding to a n-dimensional parallelotope $\mathcal{P}$. The dimension of Π is $m = n-1$.

Definition 3. *If the parallelotope $\mathcal{P}$ has zero width in direction $\mathbf{z}$ and $\mathcal{P} \oplus S(\mathbf{z})$ is an extraction of this parallelotope $\mathcal{P}$, then $\mathcal{P}$ and $\mathcal{P} \oplus S(\mathbf{z})$ determine two p-configurations denoted by Π and $\Pi \oplus H(\mathbf{z})$. The p-configuration $\Pi \oplus H(\mathbf{z})$ is called an extraction of the p-configuration Π.*

Theorem 5. *Let $\mathcal{P}$ be a parallelotope and $\mathbf{z}$ be a direction, such that $\mathcal{P}$ has zero width in direction $\mathbf{z}$. Then $\Pi \oplus H(\mathbf{z}) = \Pi \cup \Pi^{m-1}$, where Π^{m-1} is an $(m-1)$-dimensional p-configuration .*

Proof. For the segment $S(\mathbf{z})$ and an n-dimensional parallelotope $\mathcal{P}$ which has zero width in direction $\mathbf{z}$ the parallelotope $\mathcal{P} \oplus S(\mathbf{z})$ contains 1-dimensional edges $\left\langle (\mathcal{P} \oplus S(\mathbf{z}))F^1 \right\rangle$ parallel to $S(\mathbf{z})$. Using lemma 4 one can see that an $(m-1)$-dimensional hyperplane $\left\langle (\Pi \oplus H(\mathbf{z}))F^{m-1} \right\rangle$ corresponds to parallel 1-dimensional edges $\left\langle (\mathcal{P} \oplus S(\mathbf{z}))F^1 \right\rangle$ where $\left\langle (\Pi \oplus H(\mathbf{z}))F^{m-1} \right\rangle \cap (\Pi \oplus H(\mathbf{z}))$ is an $(m-1)$-dimensional p-configuration Π^{m-1}. Thus $\Pi^{m-1} \subseteq \Pi \oplus H(\mathbf{z})$ and $\Pi \subseteq \Pi \oplus H(\mathbf{z})$ consequently $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(\mathbf{z})$.

On the other hand consider an arbitrary $(k-1)$-dimensional face $(\Pi \oplus H(\mathbf{z}))F^{k-1}$ of the p-configuration $\Pi \oplus H(\mathbf{z})$. The $(k-1)$-dimensional face $(\Pi \oplus H(\mathbf{z}))F^{k-1}$ corresponds to the $(n-k)$-dimensional face $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-k}$ of the parallelotope $\mathcal{P} \oplus S(\mathbf{z})$. If the $(n-k)$-dimensional face $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-k}$ is parallel to the direction $\mathbf{z}$ then $(\Pi \oplus H(\mathbf{z}))F^{k-1} \in \Pi^{m-1}$. If the $(n-k)$-dimensional faces $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-k}$ is not parallel to the direction $\mathbf{z}$ then $(\Pi \oplus H(\mathbf{z}))F^{k-1} \in \Pi$. Consequently for an arbitrary $(k-1)$ -dimensional face $(\Pi \oplus H(\mathbf{z}))F^{k-1} \in \Pi \cup \Pi^{m-1}$. Therefore $\Pi \cup \Pi^{m-1} \supseteq \Pi \oplus H(\mathbf{z})$. By $\Pi \cup \Pi^{m-1} \subseteq \Pi \oplus H(\mathbf{z})$ and $\Pi \cup \Pi^{m-1} \supseteq \Pi \oplus H(\mathbf{z})$ the equality holds.



The above $(m-1)$-dimensional p-configuration Π^{m-1} is called a p-subconfiguration.

Definition 4. *The p-subconfigurations Π_1^{m-1} and Π_2^{m-1} are combinatorially equivalent, if $\Pi \cup \Pi_1^{m-1} \cong \Pi \cup \Pi_2^{m-1}$.*

Theorem 6. *Let $\Pi \oplus H(\mathbf{z})$ be an extraction of the p-configuration Π, then the intersection of the p-subconfiguration Π^{m-1} and an arbitrary p-line of the p-configuration Π is*

$(P1)$ *a p-line of the p-configuration Π*

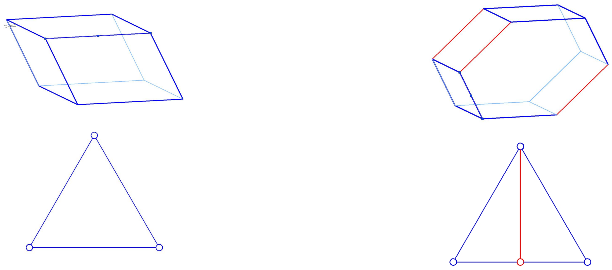
$(P2)$ *a p-vertex of the p-configuration Π,*

$(P3)$ *a p-vertex of the p-configuration $\Pi \oplus H(\mathbf{z})$, which is not a p-vertex of the p-configuration Π. In this situation the p-line has to contain two p-vertices of Π.*

Proof. Case 1. If the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ are parallel to the direction $\mathbf{z}$, then the corresponding 1-dimensional p-line $\langle \Pi F^1\rangle$ is a p-line of the p-configuration Π and of the p-subconfiguration Π^{m-1}, as well.

Case 2. If the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ are not parallel to the direction $\mathbf{z}$, then the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ intersect the shadow boundary $sh_{\mathbf{z}}(\mathcal{P})$ in two opposite facets PF_i^{n-1} or two $(n-2)$-dimensional faces $\overline{PF_i}^{n-2}$ because the shadow boundary of a convex polytope is the union of its several $(n-1)$- and $(n-2)$-dimensional closed faces. If the intersection is two facets PF_i^{n-1}, then the corresponding 1-dimensional p-line $\langle \Pi F^1\rangle$ contains the corresponding p-vertex $\langle \Pi F^0\rangle$.

Case 3. If the intersection is two $(n-2)$-dimensional faces $\overline{PF_i}^{n-2}$, then $\overline{PF_i}^{n-2} \oplus S(\mathbf{z})$ are facets $\overline{PF_i}^{n-1}$ of the parallelotope $\mathcal{P} \oplus S(\mathbf{z})$, consequently a 1-dimensional p-line $\langle \Pi F^1\rangle$ corresponding to the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ intersects the p-subconfiguration Π^{m-1} in a new p-vertex corresponding to two opposite facets $\overline{(\mathcal{P} \oplus S(\mathbf{z}))F_i}^{n-1}$ of $(\mathcal{P} \oplus S(\mathbf{z}))$. If the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ of the parallelotope $\mathcal{P}$ determine a 3-belt, then after the extraction it will be a 4-belt. Consequently the $(n-2)$-dimensional faces $\langle PF^{n-2}\rangle$ of the parallelotope $\mathcal{P}$ determine a 2-belt, thus the 1-dimensional p-line $\langle \Pi F^1\rangle$ contains two p-vertices of the p-configuration Π.



Definition 5. *A p-line of the p-configuration Π is called a p3-line if the p-line contains three p-vertices. A p-vertex of the p-configuration Π is called a p3-vertex, if the p-vertex belongs to at least one p3-line. Let H^k be a k-dimensional subspace $k < m$.*

$$V_{p3}^{k}(H^k) = \left\{ v_i | v_i \in H^k \cap \Pi, v_i \text{ a p3-vertex} \right\}, \tag{5}$$

$$L_{p3}^{k}(H^k) = \left\{ l_i | l_i \in H^k \cap \Pi, l_i \text{ a p3-line} \right\}. \tag{6}$$

Definition 6. *Let Π be a p-configuration and H^k be a k-dimensional subspace. For the set $V \subseteq V_{p3}^{k}(H^k)$ the generated set $\langle V \rangle$ is defined in the following manner:*

- *If $V = \{v_i\}$, then*

$$\langle V \rangle = \langle v_i \rangle := \{v_i\}. \tag{7}$$

- *If $V = \{v_i, v_j\}$, then*

$$\langle V \rangle = \langle v_i, v_j \rangle := \left\{ v_k | v_i, v_j, v_k \in l_i, l_i \in L_{p3}^{k}(H^k) \right\}. \tag{8}$$

- *If $|V| > 2$, then*

$$\langle V \rangle := \left\{ v_k | v_i, v_j, v_k \in l_i, l_i \in L_{p3}^{k}(H^k), v_i \in \langle V_i \rangle, v_j \in \langle V_j \rangle, V_i, V_j \subset V, V_i \cap V_j = \emptyset \right\}. \tag{9}$$

The generating system of the p3-vertices $V_{p3}^{k}(H^k)$ is $G\left(V_{p3}^{k}(H^k)\right)$, if $G\left(V_{p3}^{k}(H^k)\right) \subseteq V_{p3}^{k}(H^k)$ and $\left\langle G\left(V_{p3}^{k}(H^k)\right)\right\rangle = V_{p3}^{k}(H^k)$.

By theorem 3 every vertex v_j of the p-configuration Π belongs to at least $(n-1)$ p-lines l_i and these lines generate an $(n-1)$-dimensional hyperplane. Consequently a basis can be choose form these p-lines which is denoted by $B(L_{v_i}^{n-1}) = \{l_i | v_j \in l_i, \, i = 1, \ldots, n-1\}$.

Definition 7. *Consider a m-dimensional p-configuration Π and an $(m-1)$-dimensional hyperplane H^{m-1}. The $(m-1)$-dimensional hyperplane H^{m-1} is called a supplementer hyperplane of a p-configuration Π, if the hyperplane H^{m-1} and every p-line of the p-configuration Π has at least one common point and every p-line contains at most three p-vertices (along with the new points) and there is a generating system of p3-vertices $G\left(V_{p3}^{m-1}(H^{m-1})\right)$ for which $G\left(V_{p3}^{m-1}(H^{m-1})\right) \subseteq V(L_{v_i}^{m})$ for any $v_i \notin H^{m-1}$, where $V(L_{v_i}^{m}) = \left\{ v_k | v_k = l_i \cap H^{m-1} \text{ and } l_i \in B(L_{v_i}^{n-1}) \right\}$ and the interior of the convex hull of points $V(L_{v_i}^{m})$ does not contain any vertex of Π.*

Lemma 5. *Let $PF_{v_i}^{n-1}$ be a facet corresponding to a vertex $v_i \in \Pi$. If $PF_{v_i}^{n-1} \| Z$ for every vertex v_i of generating system $G\left(V_{p3}^{m-1}(H^{m-1})\right)$, then $PF_{v_k}^{n-1} \| Z$ for every vertex $v_k \in V_{p3}^{m-1}(H^{m-1})$, where Z is a hyperplane.*

Proof. At first, if for every vertex $v_p \in G\left(V_{p3}^{m-1}(H^{m-1})\right)$ the facet $PF_{v_p}^{n-1}$ is parallel to$Z$ for every vertex $v_p \in \langle v_p \rangle$ then for every set $\{v_m, v_n\} \subseteq G\left(V_{p3}^{m-1}(H^{m-1})\right)$ it holds that $PF_{v_q}^{n-1} \| Z$ for every vertex $v_q \in \langle v_m, v_n \rangle$ because $PF_{v_m}^{n-1} \| Z$ and $PF_{v_n}^{n-1} \| Z$ and by definition 6 $v_q, v_m, v_n \in l_i$, thus $PF_{v_q}^{n-1}, PF_{v_m}^{n-1}, PF_{v_n}^{n-1}$ are the facets of a 3-belt, consequently $PF_{v_q}^{n-1} \| Z$.

Generally suppose that for every set $V_j \subset V_i \subseteq G\left(V_{p3}^{m-1}(H^{m-1})\right)$ the facet $PF_{v_j}^{n-1}$ is parallel to Z for every vertex $v_j \in \langle V_j \rangle$. We prove that $PF_{v_i}^{n-1} \| Z$ for every vertex $v_i \in \langle V_i \rangle$. If $v_i \in \langle V_i \rangle$ according to definition 6 there is a $v_t \in \langle V_t \rangle, v_l \in \langle V_l \rangle$ such that $V_t, V_l \subset V_i$ and $V_t \cap V_l = \emptyset$, $v_t, v_l, v_i \in l_i, l_i \in L_{p3}^{k}(H^k)$, thus $PF_{v_t}^{n-1}, PF_{v_l}^{n-1}, PF_{v_i}^{n-1}$ are the facets of a 3-belt and because of the assumptions $PF_{v_t}^{n-1} \| Z$ and $PF_{v_l}^{n-1} \| Z$ the facet $PF_{v_i}^{n-1}$ is parallel to Z.

Consequently $PF_{v_k}^{n-1} \| Z$ for every vertex $v_k \in V_{p3}^{m-1}(H^{m-1}) = \left\langle G\left(V_{p3}^{m-1}(H^{m-1})\right)\right\rangle$.

Theorem 7. *If an $(m-1)$-dimensional hyperplane H^{m-1} is a supplementer hyperplane of a p-configuration Π, then there is a direction $\mathbf{z}$ that $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope and $\Pi^{m-1} \subseteq H^{m-1}$ for the p-subconfiguration Π^{m-1} of the p-configuration $\Pi \oplus H(\mathbf{z})$.*

Proof. Since $|V(L^m_{v_i})| = |B(L^m_{v_i})| = n-1$ and the hyperplane H^{m-1} is a supplementer hyperplane, $G\left(V^{m-1}_{p3}(H^{m-1})\right) \subseteq V(L^m_{v_i})$, thus $\left|G\left(V^{m-1}_{p3}(H^{m-1})\right)\right| \leq n-1$ and the number of the facets PF^{n-1}_G of $\mathcal{P}$ corresponding to the generating system of the p3-vertices $G\left(V^{m-1}_{p3}(H^{m-1})\right)$ is at most $(n-1)$, therefore the intersection of these facets is an at least 1-dimensional space Z. By lemma 5 every facet of $\mathcal{P}$ corresponding to the p3-vertices of $V^{m-1}_{p3}(H^{m-1})$ are parallel to the space Z, consequently according to theorem 2 the $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope for the segment $S(\mathbf{z}) \in Z$.

If $v_k \in V(L^m_{v_i})$ and $v_k \in \Pi$, then the facet PF^{n-1} corresponds to the p-vertex $v_k = \Pi F^0$. If $v_l \in V(L^m_{v_i})$ and $v_l \notin \Pi$ then there is a p-segment ΠF^1, for which $\Pi F^1 \cap H^{m-1} = v_l$ and the $(n-2)$-dimensional face PF^{n-2} corresponds to the p-segment ΠF^1. Consider an $(n-1)$-dimensional support hyperplane SF^{n-1} of the $(n-2)$-dimensional face PF^{n-2}. So SF^{n-1} corresponds to the p-segment ΠF^1. $B(L^{n-1}_{v_i})$ is a basis of the p-lines $l_i \notin H^{m-1}$, $i = 1, \ldots, n-1$, thus the intersection of all facets PF^{n-1}_j corresponding to $\Pi F^0_j \subset l_j \in L_j$ and all $(n-1)$-dimensional support hyperplanes SF^{n-1}_i corresponding to $\Pi F^1_i \subset l_i \in L_i$ is a line $\mathbf{z}$, where $L_i \cup L_j = B(L^m_{v_i})$. $G\left(V^{m-1}_{p3}(H^{m-1})\right) \subseteq V(L^m_{v_i})$ thus for the segment $S(\mathbf{z})$ the $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope.

The facets PF^{n-1}_j and SF^{n-1}_i are equivalent to the facets $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-1}$ thus for every $v_k \in V(L^m_{v_i})$, $v_k = (\Pi \oplus H(\mathbf{z}))F^0$. Furthermore the intersection of the facets $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-1}$ is a segment $S(\mathbf{z})$. By lemma 3 $(\Pi \oplus H(\mathbf{z}))F^{m-1}$ corresponding to the segment $S(\mathbf{z})$ contains all vertices $v_k \in V(L^m_{v_i})$, consequently $\left\langle (\Pi \oplus H(\mathbf{z}))F^{m-1} \right\rangle = H^{m-1}$ because both $(m-1)$-dimnesional hyperplanes contain the same m-dimensional simplex. By lemma 4 $\left\langle (\Pi \oplus H(\mathbf{z}))F^{m-1} \right\rangle \cap (\Pi \oplus H(\mathbf{z}))$ is an $(m-1)$-dimensional p-subconfiguration Π^{m-1}, thus $H^{m-1} \cap (\Pi \oplus H(\mathbf{z})) = \Pi^{m-1}$, consequently $\Pi^{m-1} \subseteq H^{m-1}$.

Definition 8. *Consider the supplementer hyperplane H^{m-1} of a p-configuration Π. The vertices of $\Pi \cap H^{m-1}$ are defined by points v_i for which*

$(V1)$ $v_i \in \Pi$ *and* $v_i \in H^{m-1}$,

$(V2)$ $v_i = l_i \cap H^{m-1}$ *where* $l_i \not\subseteq H^{m-1}$.

The lines of $\Pi \cap H^{m-1}$ are defined by

$(L1)$ *p-lines $l_i \in \Pi$ and $l_i \in H^{m-1}$,*

$(L2)$ *$S \cap H^{m-1}$ where S is a 2-dimensional plane generated by two p-lines l_i and l_j of the p-configuration Π and $l_i, l_j \not\subseteq H^{m-1}$.*

Theorem 8. *For the supplementer hyperplane H^{m-1} of a p-configuration Π the intersection $\Pi \cap H^{m-1}$ is equal to the p-subconfiguration Π^{m-1} of the p-configuration $\Pi \oplus H(\mathbf{z})$.*

Proof. By theorem 7, if an $(m-1)$-dimensional hyperplane H^{m-1} is a supplementer hyperplane of Π, then there is a direction $\mathbf{z}$ such that $\mathcal{P} \oplus S(\mathbf{z})$ is a parallelotope and $\Pi^{m-1} \subseteq H^{m-1}$ for the p-subconfiguration Π^{m-1} of $\Pi \oplus H(\mathbf{z})$. According to theorem 6 and definition 8 $(P2)$ is equal to $(V1)$, $(P3)$ is equal to $(V2)$ and $(P1)$ is equal to $(L1)$, consequently it is sufficient to prove the equality for case $(L2)$.

Let l be equal to $S \cap H^{m-1}$. By the case $(L2)$ of the definition 8 p-lines l_i and l_j generate a 2-dimensional plane S. On the one hand we can choose l_i, l_j in such a way that the p-segments $\Pi F^1_i \subseteq l_i$ and $\Pi F^1_j \subseteq l_j$ have a common polygon $\Pi F^2 \subseteq S$. On the other hand $l_i, l_j \subseteq S$ and $l_i, l_j \not\subseteq H^{m-1}$, consequently $l_i, l_j \not\subseteq \Pi^{m-1}$ by theorem 7. Thus $l_i \cap \Pi^{m-1} = v_i$ and $l_j \cap \Pi^{m-1} = v_j$ are p-vertices of the subconfiguration Π^{m-1} according to case $(P3)$ of theorem 6.

The polygon ΠF^2 corresponds to an $(n-3)$-dimensional face PF^{n-3}, the p-segments ΠF^1_i and ΠF^1_j correspond to the $(n-2)$-dimensional faces PF^{n-2}_i, PF^{n-2}_j, where $\Pi F^1_i \subseteq \Pi F^2$, $\Pi F^1_j \subseteq \Pi F^2$, therefore $PF^{n-3} \subseteq PF_i^{n-2}$ and $PF^{n-3} \subseteq PF_j^{n-2}$. Thus after the extraction $PF^{n-3} \oplus S(\mathbf{z}) \subseteq PF_i^{n-2} \oplus S(\mathbf{z})$ and $PF^{n-3} \oplus S(\mathbf{z}) \subseteq PF_j^{n-2} \oplus S(\mathbf{z})$, where p-vertices v_i and v_j correspond to the facets $PF_i^{n-2} \oplus S(\mathbf{z})$ and $PF_j^{n-2} \oplus S(\mathbf{z})$ and a p-line $\bar{l}$ corresponds to the $(n-2)$-dimensional faces $\langle PF^{n-3} \oplus S(\mathbf{z}) \rangle$ of $\mathcal{P} \oplus S(\mathbf{z})$. By lemma 2 $v_i, v_j \in \bar{l}$, consequently $\bar{l} = l$. That is $l \notin \Pi$ and $l \in \Pi \oplus H(\mathbf{z})$, thus $l \in \Pi^{m-1}$.

Consider a p-line l for which $l \notin \Pi$ and $l \in \Pi^{m-1}$. Let H^{m-1} be an $(m-1)$-dimensional hyperplane of the p-subconfiguration Π^{m-1}. Thus $\Pi^{m-1} \subseteq H^{m-1}$, consequently $l \subseteq H^{m-1}$. If the segment $(\Pi \oplus H(\mathbf{z}))F^1 \subseteq l$ and

$$(\Pi \oplus H(\mathbf{z}))F_1^0 \subseteq (\Pi \oplus H(\mathbf{z}))F^1 \text{ and } (\Pi \oplus H(\mathbf{z}))F_2^0 \subseteq (\Pi \oplus H(\mathbf{z}))F^1, \tag{10}$$

then for the corresponding parallelotope $\mathcal{P} \oplus S(\mathbf{z})$ by lemma 2

$$(\mathcal{P} \oplus S(\mathbf{z}))F_1^{n-1} \supseteq (\mathcal{P} \oplus S(\mathbf{z}))F^{n-2} \text{ and } (\mathcal{P} \oplus S(\mathbf{z}))F_2^{n-1} \supseteq (\mathcal{P} \oplus S(\mathbf{z}))F^{n-2}. \tag{11}$$

Before the extraction of the parallelotope $\mathcal{P}$ the $(n-2)$-dimensional face $(\mathcal{P} \oplus S(\mathbf{z}))F^{n-2}$ was $(n-3)$-dimensional face $\mathcal{P}F^{n-3}$. For the facets $(\mathcal{P} \oplus S(\mathbf{z}))F_1^{n-1}, (\mathcal{P} \oplus S(\mathbf{z}))F_2^{n-1}$ there are two cases.

The facets $(\mathcal{P} \oplus S(\mathbf{z}))F_1^{n-1}, (\mathcal{P} \oplus S(\mathbf{z}))F_2^{n-1}$ are either $(n-2)$-dimensional faces $\mathcal{P}F_1^{n-2}, \mathcal{P}F_2^{n-2}$ or facets $\mathcal{P}F_1^{n-1}, \mathcal{P}F_2^{n-1}$. In the second case we choose $(n-2)$-dimensional faces $\mathcal{P}F_1^{n-2} \subseteq \mathcal{P}F_1^{n-1}$ and $\mathcal{P}F_2^{n-2} \subseteq \mathcal{P}F_2^{n-1}$ for which

$$\mathcal{P}F_1^{n-2} \supseteq \mathcal{P}F^{n-3} \text{ and } \mathcal{P}F_2^{n-2} \supseteq \mathcal{P}F^{n-3}. \tag{12}$$

In the first case this relation holds. Thus for the p-configuration Π by lemma 2

$$\Pi F_1^1 \subseteq \Pi F^2 \text{ and } \Pi F_2^1 \subseteq \Pi F^2. \tag{13}$$

$S \cap H^{m-1} = l$ is satisfied for $l_1 = \langle \Pi F_1^1 \rangle$, $l_2 = \langle \Pi F_2^1 \rangle$ and $S = \langle \Pi F^2 \rangle$. Q.e.d.

Consequently the p-configuration $\Pi \oplus H(\mathbf{z})$ is equal to $\Pi \cup \big(\Pi \cap H^{m-1}\big)$.

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