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SPLIT-STEP OPERATORS IN ABSOLUTE COORDINATES FOR AN EFSSM SOLUTION METHOD OF THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION

Görbe, Mihály [1*]

[1] Department of Science, GAMF Faculty of Engineering and Computer Science, John von Neumann University, Hungary
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Abstract
Exponential Fourier split-step methods (EFSSM) are widely employed nowadays in the numerical treatment of the generalized nonlinear Schrödinger equation, e.g. in the modeling of ultrashort light pulse propagation in nonlinear media. The derivation of such methods is straightforward for a normalized electric field wavefunction using normalized coordinates, i.e. spatial units that depend on diffraction length and pulse length and temporal units that depend on pulse duration, and they may depend on several more characteristic measures, such as nonlinear length, dispersion length, peak electric field strength as well. However when testing simulation programs and comparing outputs with experimental data it is easier to use absolute coordinates and absolute electric fields. This paper derives the formulae of the EFSSM in recent paper [1] using absolute coordinates for absolute electric fields.

1 Introduction

The generalized nonlinear Schrödinger equation is the model of several natural phenomena, such as plasma soliton [2] and water wave propagation [3,4], and the topic of the current paper: the propagation of ultra-intense laser pulses in nonlinear media [5,6], and its special case, the generation of white light continuum from ultra-intense laser pulses.

The white light continuum, which is an effect of spectral broadening of ultra-intense laser pulses, is illustrated on Fig. 1.

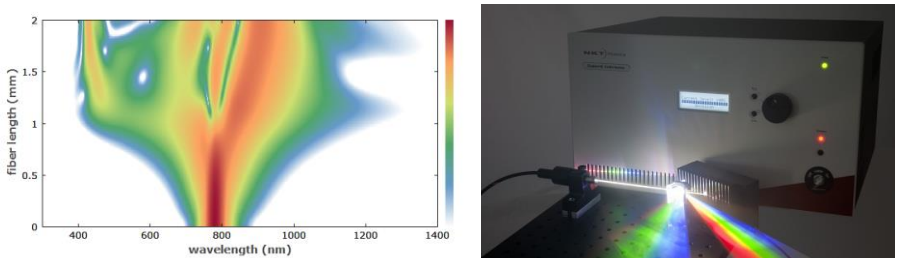
(a)  (b)

Figure 1. The input high intensity laser pulse represented at the bottom of graph (a) [7] has a narrow wavelength range, but as it propagates in a medium, it gets broader and broader, and this

* Corresponding author. Tel.: +36 76 516 436; fax: +36 76 516 399
E-mail address: gorbe.mihaly@gamf.uni-neumann.hu

broadness in the end can be seen even with the naked eye as the white color of the output light on picture (b) [7]. The white light is also resolved into the colors of the rainbow.

There are several advanced applications of the white light continuum, including optical coherence tomography [8,9], frequency metrology [10,11,12] and fluorescence lifetime imaging [13]. Of course, the extreme broad bandwidth of the white light continuum enables it to be very fast light source of optical communication [14,15,16], and it has advantages as the light source of photoacoustic gas sensing applications [17,18,19].

Modelling of white light continuum generation is therefore of great scientific and technical importance.

2 Exponential Fourier split-step method (EFSSM) for the solution of the generalized nonlinear Schrödinger equation

The form of the generalized nonlinear Schrödinger equation used in nonlinear optics for the cases without ionization is [1]

$$\frac{\partial u}{\partial \zeta} = \frac{i}{4}\left(1 + \frac{i}{\omega_0 \tau_p}\frac{\partial}{\partial \tau}\right)^{-1}\left(\nabla_\chi^2 + \nabla_\psi^2\right)u - i\frac{L_{df}}{L_{ds}}\frac{\partial^2 u}{\partial \tau^2} + i\left(1 + \frac{i}{\omega_0 \tau_p}\frac{\partial}{\partial \tau}\right)\frac{L_{df}}{L_{nl}}|u|^2 u \tag{1}$$

where $u = u(\chi,\psi,\zeta,\tau)$ is the normalized space and time dependent evolving wavefunction of the light pulse to be determined, u_0 is the normalized wavefunction of the input pulse. The coefficients are related to the linear and nonlinear optical properties of the pulse and the medium (ω_0: central angular frequency, τ_p: pulse e^{-1} length, L_{df}: diffraction length, L_{ds}: dispersion length, L_{nl}: nonlinear length). On the left hand side there is the derivative of u by ζ, which is the normalized coordinate in the direction of propagation. The Laplace operators are computed in the directions perpendicular to the propagation, with normalized χ and ψ variables. The temporal variable τ, is normalized too.

The numerical solution of this equation is possible using an exponential Fourier split-step method (EFSSM) [1]. The method takes steps in the ζ direction in order to compute $u(\zeta+\Delta\zeta)$ from $u(\zeta)$. The step is split because of the handling of the terms of the right hand side of the equation separately, for example:

$$u(\zeta + \Delta\zeta) \approx E_C\left[\Delta\zeta, \zeta, E_B\left[\Delta\zeta, \zeta, E_A[\Delta\zeta, \zeta, u(\zeta)]\right]\right]$$

where E_A, E_B and E_C are the solution operators of the differential equations having the following A, B, and C operators (so called split step operators) on their right hand sides:

$$A = \frac{i}{4}\left(1 + \frac{i}{\omega_0 \tau_p}\frac{\partial}{\partial \tau}\right)^{-1}\left(\nabla_\chi^2 + \nabla_\psi^2\right) - i\frac{L_{df}}{L_{ds}}\frac{\partial^2}{\partial \tau^2}$$
$$B = i\left[\frac{L_{df}}{L_{nl}}|u|^2\right] - \frac{1}{\omega_0 \tau_p}\left[\frac{L_{df}}{L_{nl}}\frac{\partial}{\partial \tau}|u|^2\right]$$
$$C = -\frac{1}{\omega_0 \tau_p}\left[\frac{L_{df}}{L_{nl}}|u|^2\frac{\partial}{\partial \tau}\right]$$

Here $(A + B + C)u$ is just the right hand side of our original (1) equation.

The ordering of the solution operators in step splitting, and the step sizes influence the precision of the solution. The best step size-operator ordering pattern is [1]

$$u(\zeta + \Delta\zeta) \approx$$
$$\approx E_A\left[\frac{\Delta\zeta}{4}, \zeta + \frac{3}{4}\Delta\zeta, E_c\left[\frac{\Delta\zeta}{2}, \zeta, E_A\left[\frac{\Delta\zeta}{4}, \zeta + \frac{2}{4}\Delta\zeta, E_B\left[\Delta\zeta, \zeta, E_A\left[\frac{\Delta\zeta}{4}, \zeta + \frac{1}{4}\Delta\zeta, E_c\left[\frac{\Delta\zeta}{2}, \zeta, E_A\left[\frac{\Delta\zeta}{4}, \zeta, u(\zeta)\right]\right]\right]\right]\right]\right]\right]$$

3 Motivation

In order to easily compare the EFSSM solution to experimental data or simulation results from another numerical methods, electric field strength as the function of absolute spatial and temporal coordinates $\mathcal{E}(x,y,z,t)$ should be obtained. In [1] we are given the split-step operators and the full EFSSM method for a normalized wavefunction u in normalized coordinates, and the situation is further complicated by the presence of the diffraction length, dispersion length, nonlinear length measures, that depend on both medium and pulse properties. Therefore it is straightforward to derive the corresponding operators in absolute space and time coordinates, that operators give the absolute electric field strength function.

4 Derivation

To derive the operators outlined above, first I substituted the definition of the relative coordinates into (1): $\chi = \frac{x}{s_p}$, $\psi = \frac{y}{s_p}$, $\zeta = \frac{z}{L_{df}}$ and $\tau = \frac{t}{\tau_p}$ (the latter definition is not present in [1]; s_p: pulse e^{-1} spatial extent).

$$L_{df}\frac{\partial u}{\partial z} = \frac{i}{4}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1} s_p^2\left(\nabla_x^2+\nabla_y^2\right)u - i\frac{L_{df}}{L_{ds}}\tau_p^2\frac{\partial^2 u}{\partial t^2} + i\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\frac{L_{df}}{L_{nl}}|u|^2u$$

Now substitute the definitions of the diffraction length, dispersion length and the nonlinear length: $L_{df} = \frac{k_0 s_p^2}{2}$, $L_{ds} = \frac{2\tau_p^2}{k''}$ and $L_{nl} = \frac{c}{\omega_0 n_2 I_0}$ (k_0: central wavenumber, k'': the group delay dispersion and n_2: the nonlinear refractive index of the medium, I_0: the peak intensity of the pulse), in order to separate the measures that characterize the pulse, and that characterize the medium.

$$\frac{k_0 s_p^2}{2}\frac{\partial u}{\partial z} = \frac{i}{4}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1} s_p^2\left(\nabla_x^2+\nabla_y^2\right)u - i\frac{\frac{k_0 s_p^2}{2}}{\frac{2\tau_p^2}{k''}}\tau_p^2\frac{\partial^2 u}{\partial t^2} + i\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\frac{\frac{k_0 s_p^2}{2}}{\frac{c}{\omega_0 n_2 I_0}}|u|^2u$$

Let's divide by k_0, and use the several possibilities to simplify:

$$\frac{\partial u}{\partial z} = \frac{i}{2k_0}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1}\left(\nabla_x^2+\nabla_y^2\right)u - i\frac{k''}{2}\frac{\partial^2 u}{\partial t^2} + i\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\frac{\omega_0 n_2 I_0}{c}|u|^2u \tag{2}$$

At this point it is useful to compare the results with another paper [20], where the pulse propagation equation reads:

$$\frac{\partial A}{\partial z} = \frac{i}{2k_0}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1}\left(\nabla_x^2+\nabla_y^2\right)A - \frac{i}{2}k''\frac{\partial^2 A}{\partial t^2} + \frac{i\omega_0 n_2}{c}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)|A|^2A$$

The only difference is the definition of the wavefunction, $A = \sqrt{I_0}u$. This definition makes it possible, to eliminate I_0 in the last term, by absorbing it into the square term as $\left|\sqrt{I_0}u\right|^2 = |A|^2$.

A is proportional to the electric field strength amplitude $\mathcal{E}$ we are looking for. The latter is known to be in the form $I = \frac{1}{2}\epsilon_0 cn|\mathcal{E}|^2$. The peak intensity is associated with the peak electric field amplitude: $I_0 = \frac{1}{2}\epsilon_0 cn|\mathcal{E}_0|^2$. Substitute this into the (2):

$$\frac{\partial u}{\partial z} = \frac{i}{2k_0}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1}\left(\nabla_x^2+\nabla_y^2\right)u - i\frac{k''}{2}\frac{\partial^2 u}{\partial t^2} + i\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\frac{\omega_0 n_2 \frac{1}{2}\epsilon_0 cn}{c}|\mathcal{E}_0\cdot u|^2u$$

The electric field amplitude $\mathcal{E} = \mathcal{E}_0\cdot u$ can be obtained everywhere in the equation through multiplying by $\mathcal{E}_0$:

$$\frac{\partial \mathcal{E}}{\partial z}=\frac{i}{2k_0}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1}\left(\nabla_x^2+\nabla_y^2\right)\mathcal{E}-i\frac{k"}{2}\frac{\partial^2\mathcal{E}}{\partial t^2}+i\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\omega_0 n_2\frac{1}{2}\epsilon_0 n|\mathcal{E}|^2\mathcal{E} \tag{3}$$

Now the A', B' and C' operators of (3) can be constructed, whose sum applied on $\mathcal{E}$ gives the right hand side of (3):

$$A'(x,y,t)=\frac{i}{2k_0}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^{-1}\left(\nabla_x^2+\nabla_y^2\right)-\frac{i}{2}k"\frac{\partial^2}{\partial t^2}$$
$$B'(x,y,t)=\omega_0 n_2\frac{1}{2}\epsilon_0 n\left[i|\mathcal{E}|^2-\frac{1}{\omega_0}\frac{\partial}{\partial t}|\mathcal{E}|^2\right]$$
$$C'(x,y,t)=-n_2\frac{1}{2}\epsilon_0 n|\mathcal{E}|^2\frac{\partial}{\partial t}$$

5 Results and future work

With my colleague Balázs Antalicz at ELI-ALPS we have written a modeling program in Matlab, that implements the three-operator method derived above, with the step size-operator ordering pattern:

$$\mathcal{E}(z+\Delta z)\approx$$
$$\approx E_{A'}\left[\frac{\Delta z}{4},z+\frac{3}{4}\Delta z,E_{C'}\left[\frac{\Delta z}{2},z,E_{A'}\left[\frac{\Delta z}{4},z+\frac{2}{4}\Delta z,E_{B'}\left[\Delta z,z,E_{A'}\left[\frac{\Delta z}{4},z+\frac{1}{4}\Delta z,E_{C'}\left[\frac{\Delta z}{2},z,E_{A'}\left[\frac{\Delta z}{4},z,\mathcal{E}(z)\right]\right]\right]\right]\right]\right]\right]$$

A few examples I generated using this code can be seen on Fig. 2. An ultra-intense laser pulse is taken into account propagating in fused silica (for quantitative properties see figure caption). The horizontal extent is the spatial size, the vertical is the spectrum, that broadens spectacularly as the pulse propagates forward.

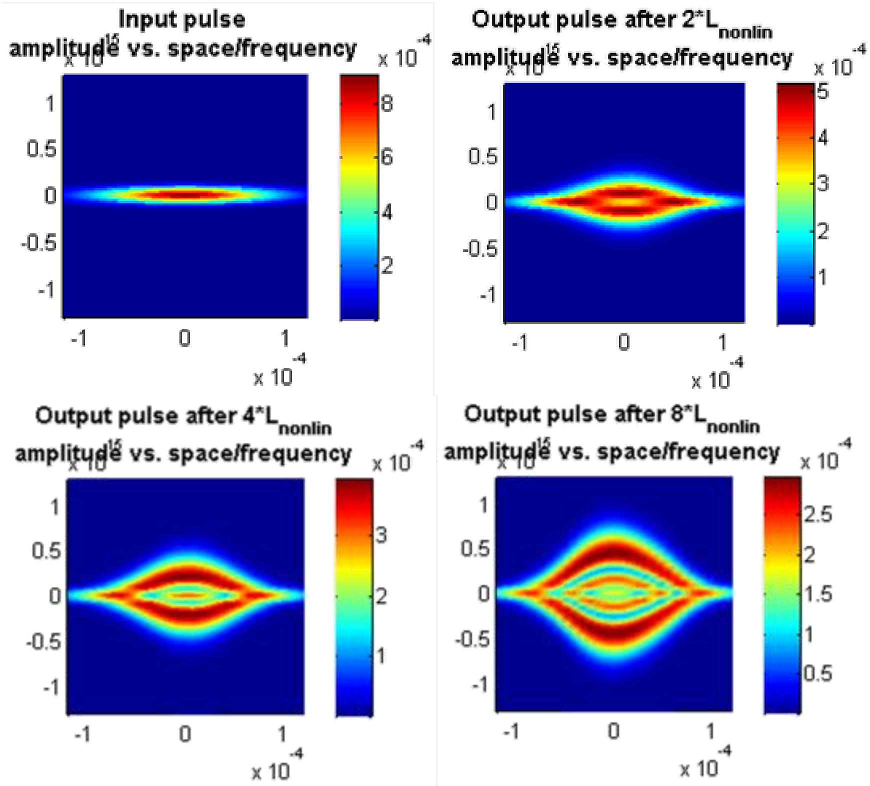


Figure 2. Ultra-intense light pulse propagation in fused silica. Central wavelength: 532 nm, input length: 30 fs, beam width: 100 µm, pulse energy: 200 mJ (peak field strength $2{\cdot}10^{10}$ V/m)

My planned future task is to recreate the multiplate continuum-generation experiments in [20] as simulations with an improved code, and then to investigate the effect of different types and extents of spatio-temporal couplings of the pulse on the output white light continuum.

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